Pattern avoiding 3-permutations and triangle bases

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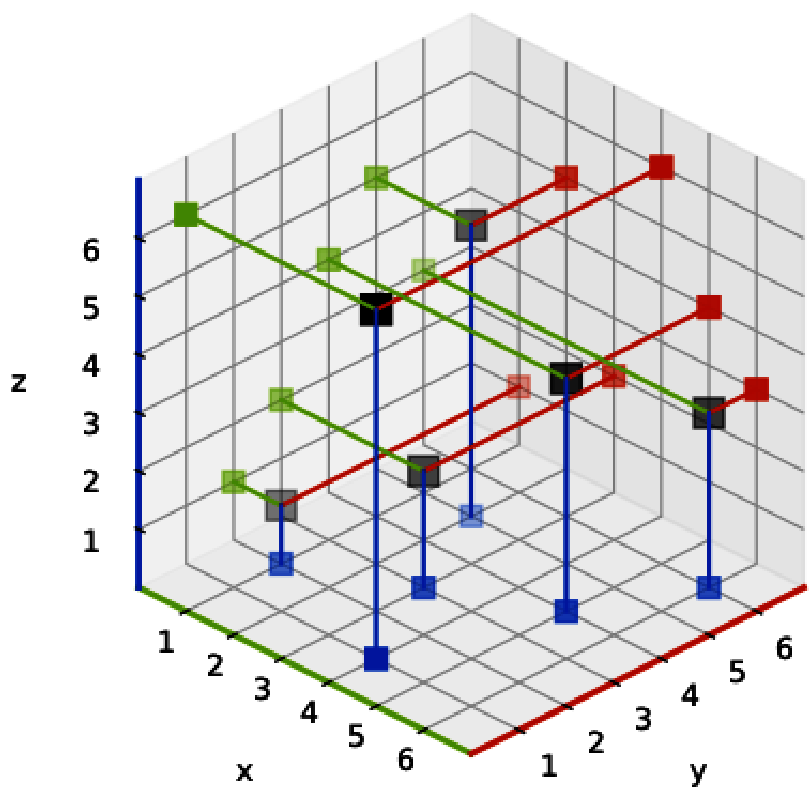
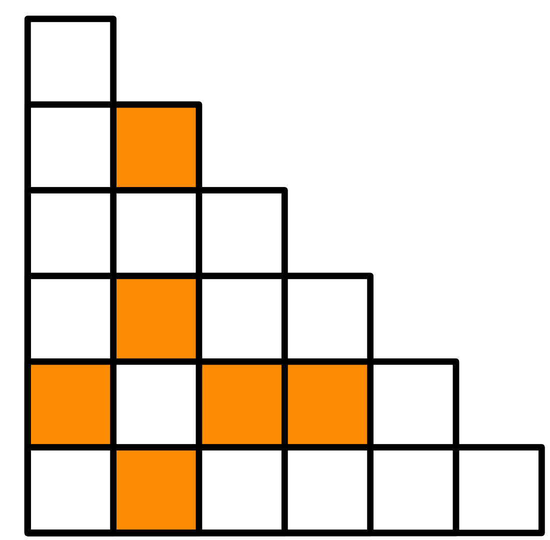
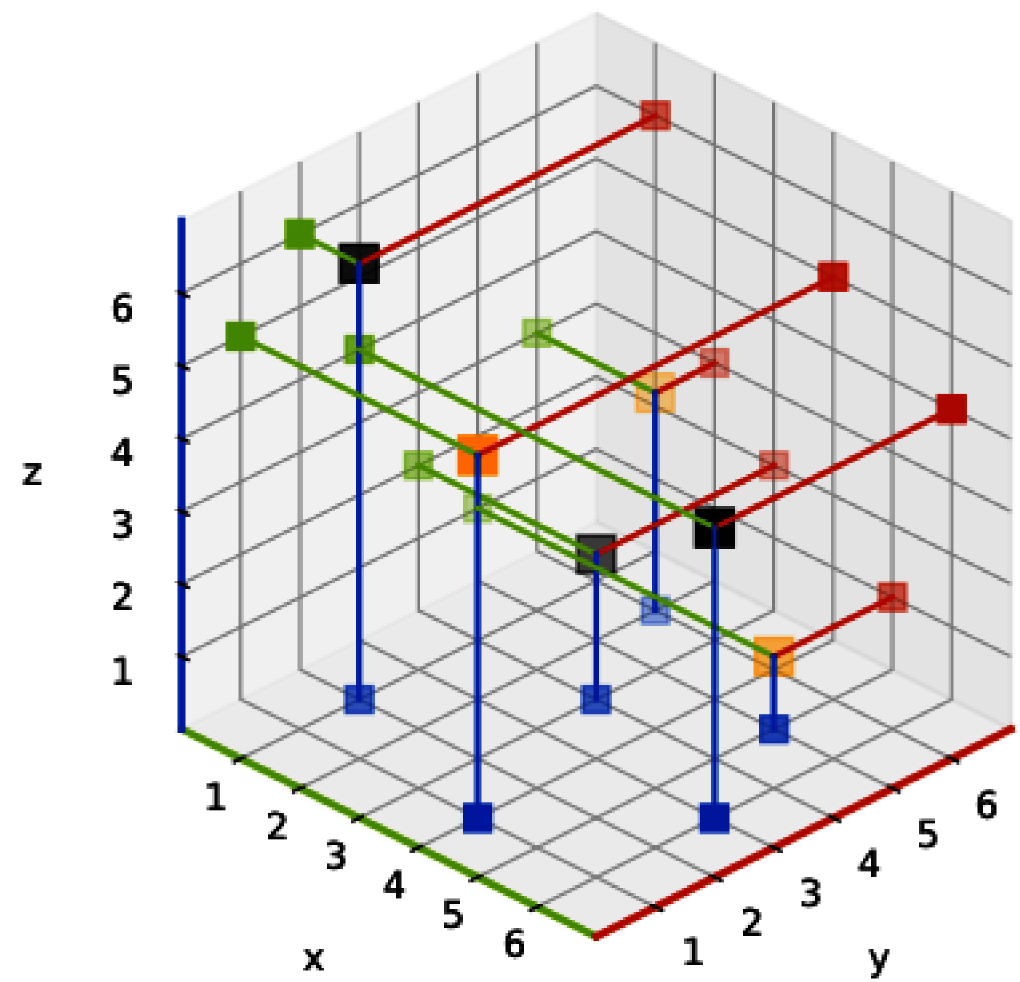




I- The objects

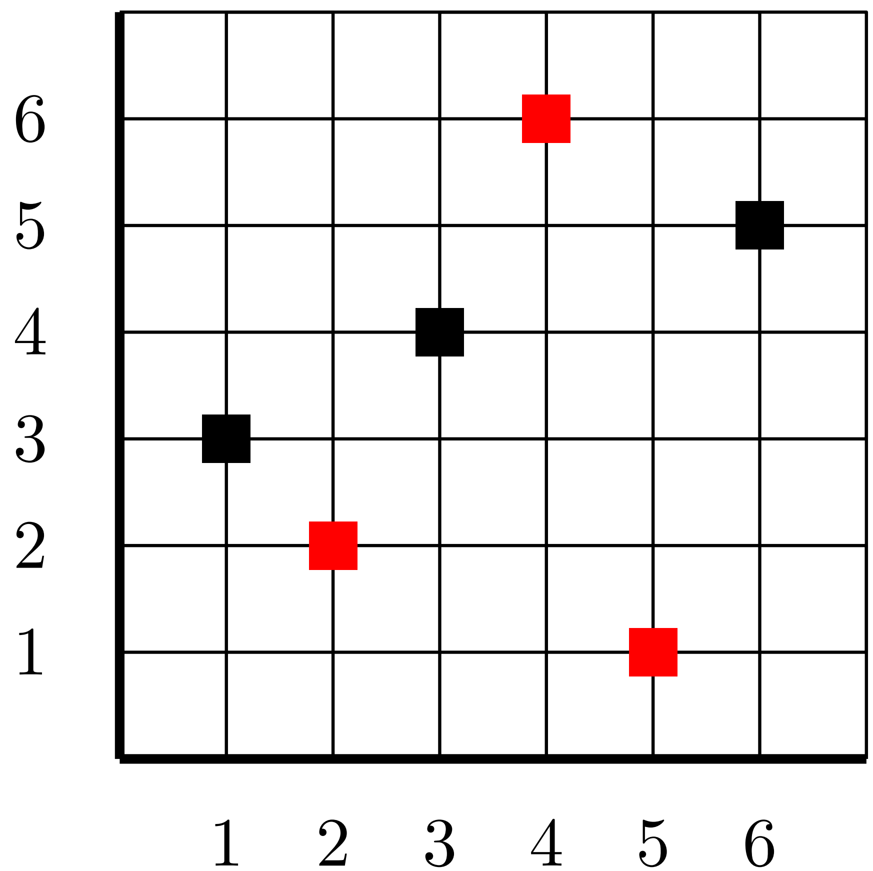
a) Pattern avoiding 3-permutations

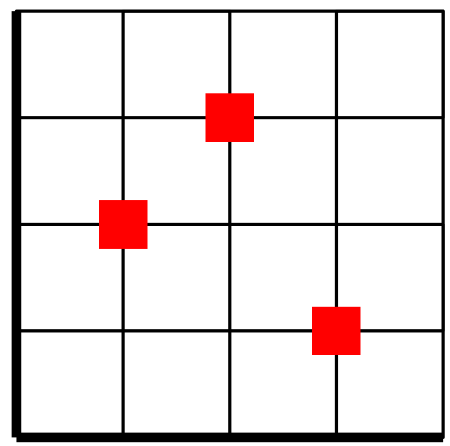


Pattern avoidance in permutations

The **diagram** of a permutation $\sigma \in \mathfrak{S}_n$ is the set of points $P_\sigma = \{(i, \sigma(i)) \mid 1 \leqslant i \leqslant n\}$. It has exactly one point per row and per column.

A permutation $\sigma \in \mathfrak{S}_n$ **contains** a pattern $\pi \in \mathfrak{S}_k$ if there is a set of indices I such that $\sigma_{|I} \simeq \pi$. Otherwise, it **avoids** it.





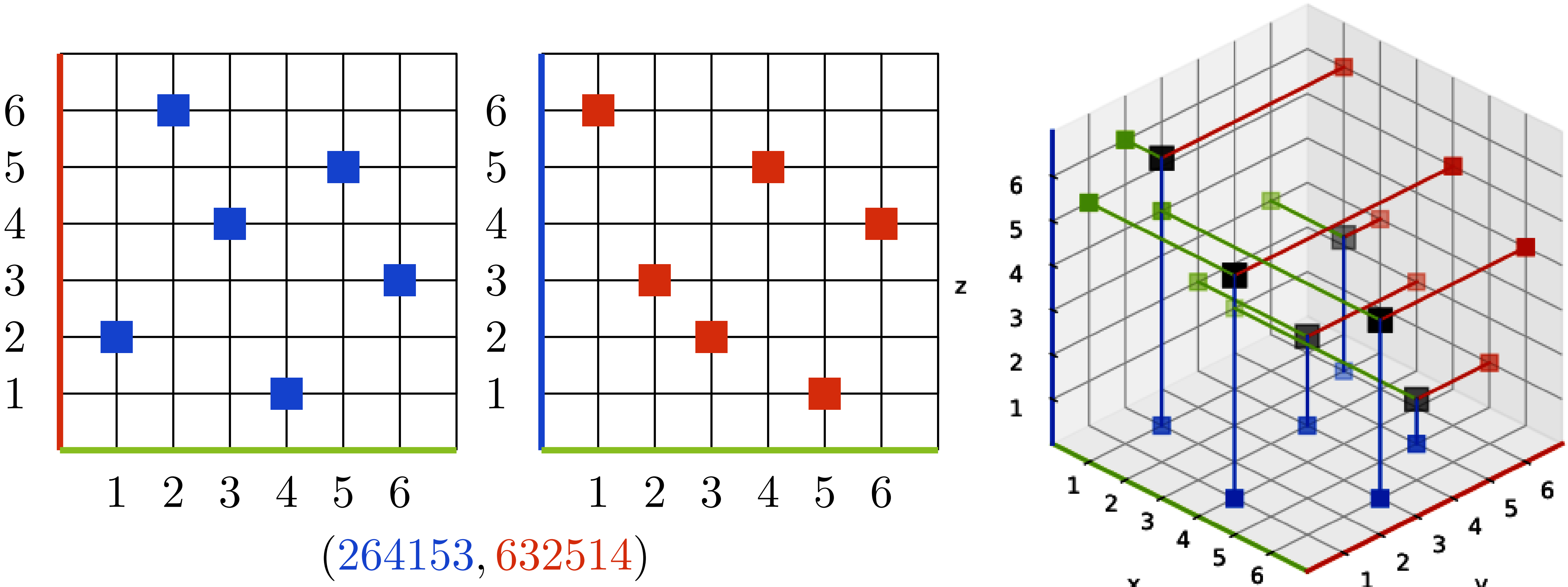
$\sigma = 324615$ contains the pattern $\pi = 231$.

Pattern avoidance in 3-permutations

A **3-diagram** has exactly one point per plane of the grid.

It is coded by a **3-permutation** $(\sigma, \tau) \in \mathfrak{S}_n^2$:

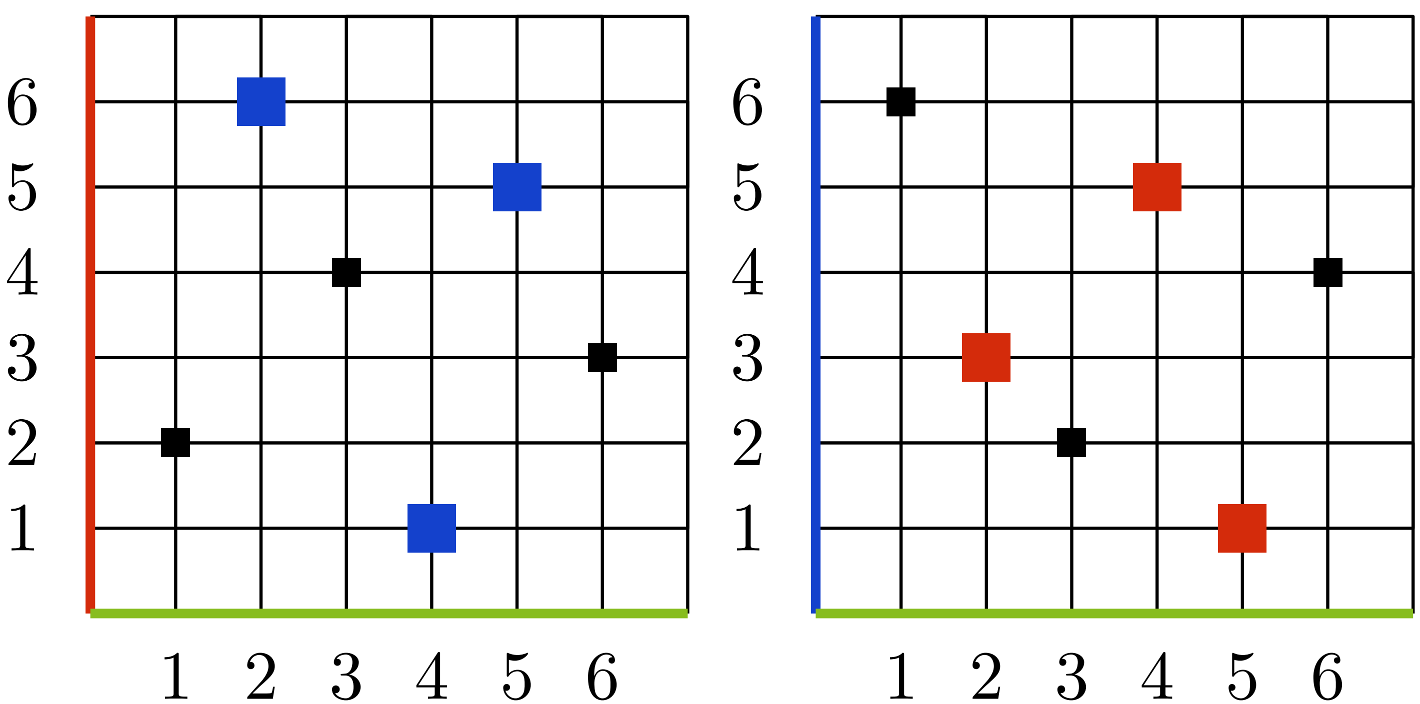
$$P_{(\sigma,\tau)} = \{(i, \sigma(i), \tau(i)) \mid 1 \leqslant i \leqslant n\}.$$



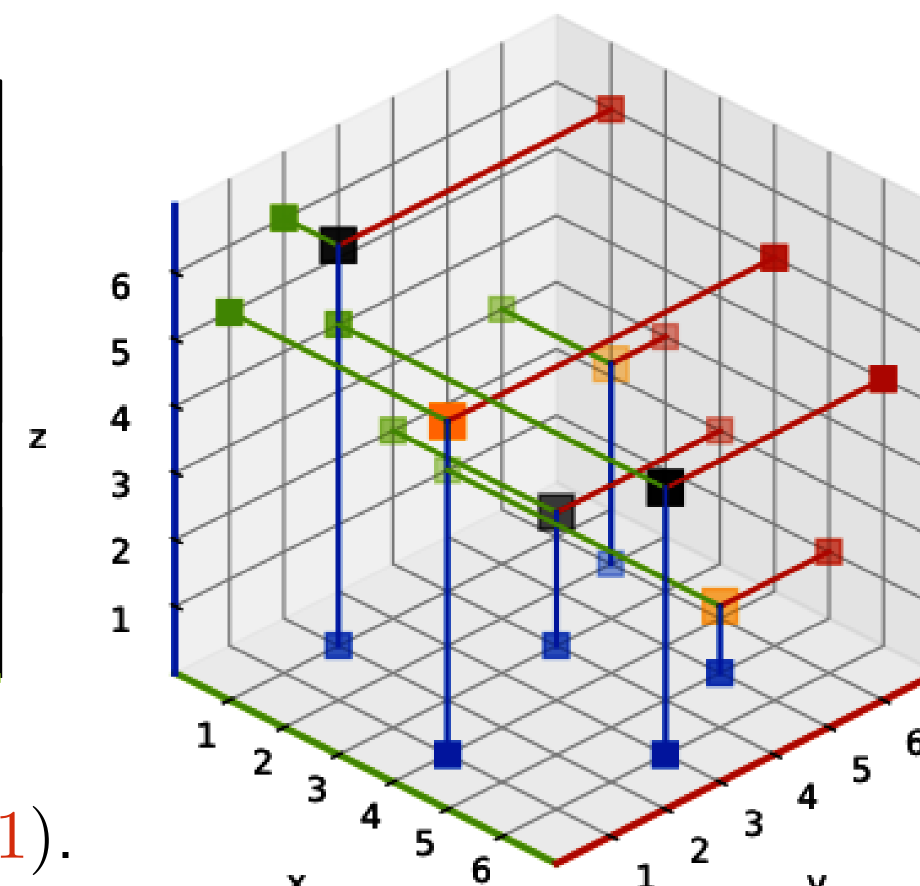
Pattern avoidance in 3-permutations

A 3-**diagram** has exactly one point per plane of the grid.

A 3-permutation $(\sigma, \tau) \in \mathfrak{S}_n^2$ **contains** a pattern $(\pi_1, \pi_2) \in \mathfrak{S}_k^2$ if there is a set of indices $I \subset [\![1, n]\!]$ such that $\sigma_{|I} \simeq \pi_1$ and $\tau_{|I} \simeq \pi_2$.
Otherwise it **avoids** it.



$(264153, 632514)$ contains the pattern $(312, 231)$.



Pattern avoidance classes

Patterns	TWE	Sequence	Comment
$(12, 12)$	4	$1, 3, 17, 151, 1899, 31711, \cdots$	weak-Bruhat intervals
$(12, 12), (12, 21)$	6	$n! = 1, 2, 6, 24, 120 \cdots$	$\sigma_1 \Rightarrow \sigma_2$
$(12, 12)$, $(12, 21)$, $(21, 12)$	4	$1, 1, 1, 1, 1, 1, \cdots$	1 diagonal
$(12, 12)$, $(12, 21)$, $(21, 12)$, $(21, 21)$	1	$1, 0, 0, 0, 0, 0, \cdots$	
$(123, 123)$	4	$1, 4, 35, 524, 11774, 366352, \cdots$	*new*
$(123, 132)$	24	$1, 4, 35, 524, 11768, 365558, \cdots$	*new*
$(132, 213)$	8	$1, 4, 35, 524, 11759, 364372, \cdots$	*new*
$(12, 12), (132, 312)$	48	$(n+1)^{n-1} = 1, 3, 16, 125, 1296 \cdots$	[Atkinson et al. 93,95]
$(12, 12), (123, 321)$	12	$1, 3, 16, 124, 1262, 15898, \cdots$	distributive lattices inter.
$(12, 12), (231, 312)$	8	$1, 3, 16, 122, 1188, 13844, \cdots$	*A295928?*

[Bonichon & Morel '22]

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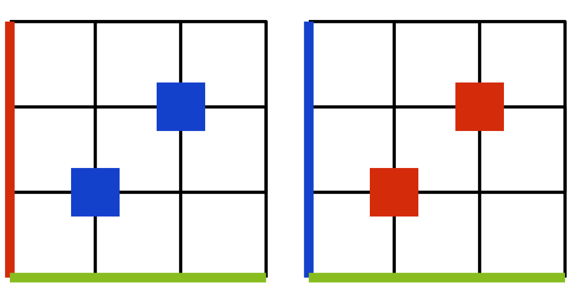
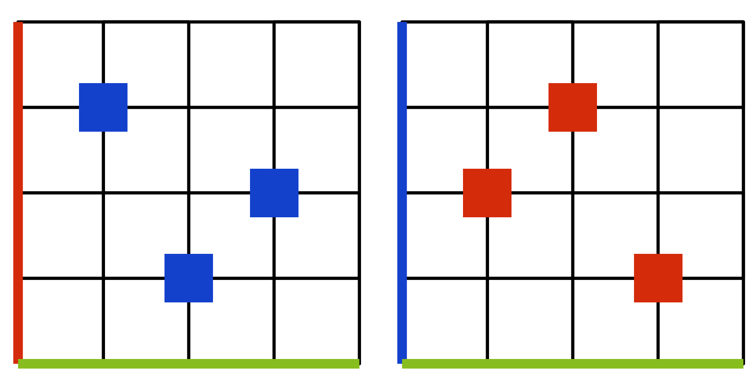
[Bonichon & Morel '22]



$(12, 12)$



$(312, 231)$

Pattern avoidance classes

A295928 Number of triangular matrices T(n,i,k), k <= i <= n, with entries "0" or "1" with the property that each triple {T(n,i,k), T(n,i,k+1), T(n,i-1,k)} containing a single "0" can be successively replaced by {1, 1, 1} until finally no "0" entry remains. 1

1, 3, 16, 122, 1188, 13844, 185448, 2781348, 45868268

(list; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS A triple {T(n,i,k), T(n,i,k+1), T(n,i-1,k)} will be called a primitive triangle. It is easy to see that b(n) = n(n-1)/2 is the number of such triangles. At each step, exactly one primitive triangle is completed (replaced by {1, 1, 1}). So there are b(n) "0"- and n "1"-terms. Thus the starting matrix has no complete primitive triangle. Furthermore, any triangular submatrix T(m,i,k), k <= i <= m < n cannot have more than m "1"-terms because otherwise it would have less "0"-terms than primitive triangles. The replacement of at least one "0"-term would complete more than one primitive triangle. This has been excluded.

So T(n, i, k) is a special case of U(n, i, k), described in A101481: a(n) < A101481(n+1).

A start matrix may serve as a pattern for a number wall used on worksheets for elementary mathematics, see link "Number walls". That is why I prefer the more descriptive name "fill matrix".

The algorithm for the sequence is rather slow because each start matrix is constructed separately. There exists a faster recursive algorithm which produces the same terms and therefore is likely to be correct, but it is based on a conjecture. For the theory of the recurrence, see "Recursive aspects of fill matrices". Probable extension a(10)-a(14): 821096828, 15804092592, 324709899276, 7081361097108, 163179784397820.

The number of fill matrices with n rows and all "1"- terms concentrated on the last two rows, is A001960(n). See link "Reconstruction of a sequence".

LINKS Table of n, a(n) for n=1..9.

Gerhard Kirchner, Recursive aspects of fill matrices

Gerhard Kirchner, Number walls

Gerhard Kirchner, VB-program

Gerhard Kirchner, Reconstruction of a sequence

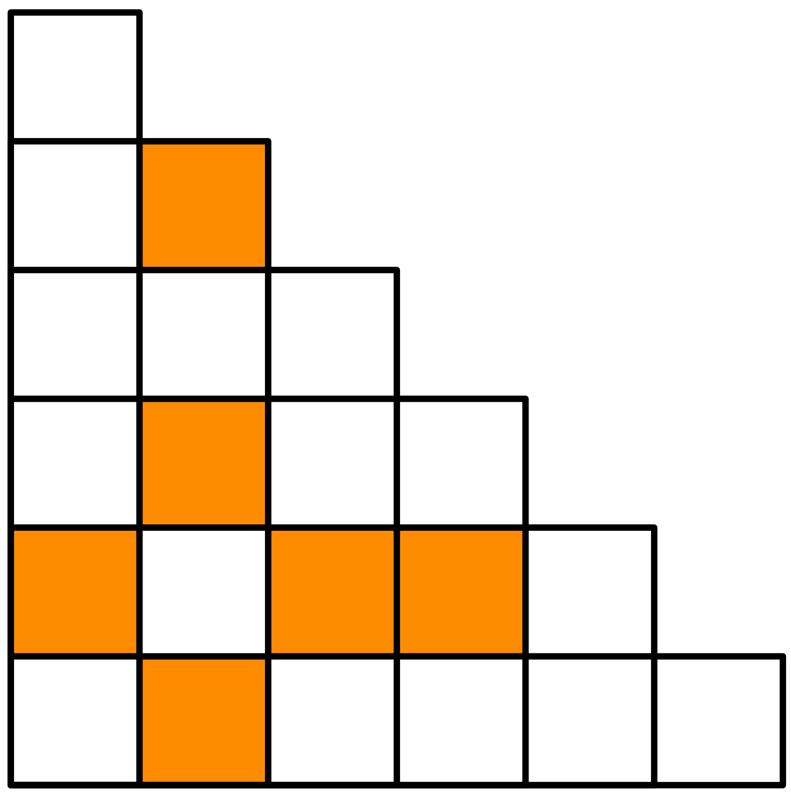
Ville Salo, Cutting Corners, arXiv:2002.08730 [math.DS], 2020.

Yuan Yao and Fedir Yudin, Fine Mixed Subdivisions of a Dilated Triangle, arXiv:2402.13342 [math.CO], 2024.

EXAMPLE
```
Example (n=2):    0     1    1
    a(2)=3       1 1   0 1  1 0
Example for completing a 3-matrix (3 bottom terms):
    1        1       1       1
   0 0  ->  1 0 ->  1 1 ->  1 1
  1 1 0    1 1 0   1 1 0   1 1 1
```

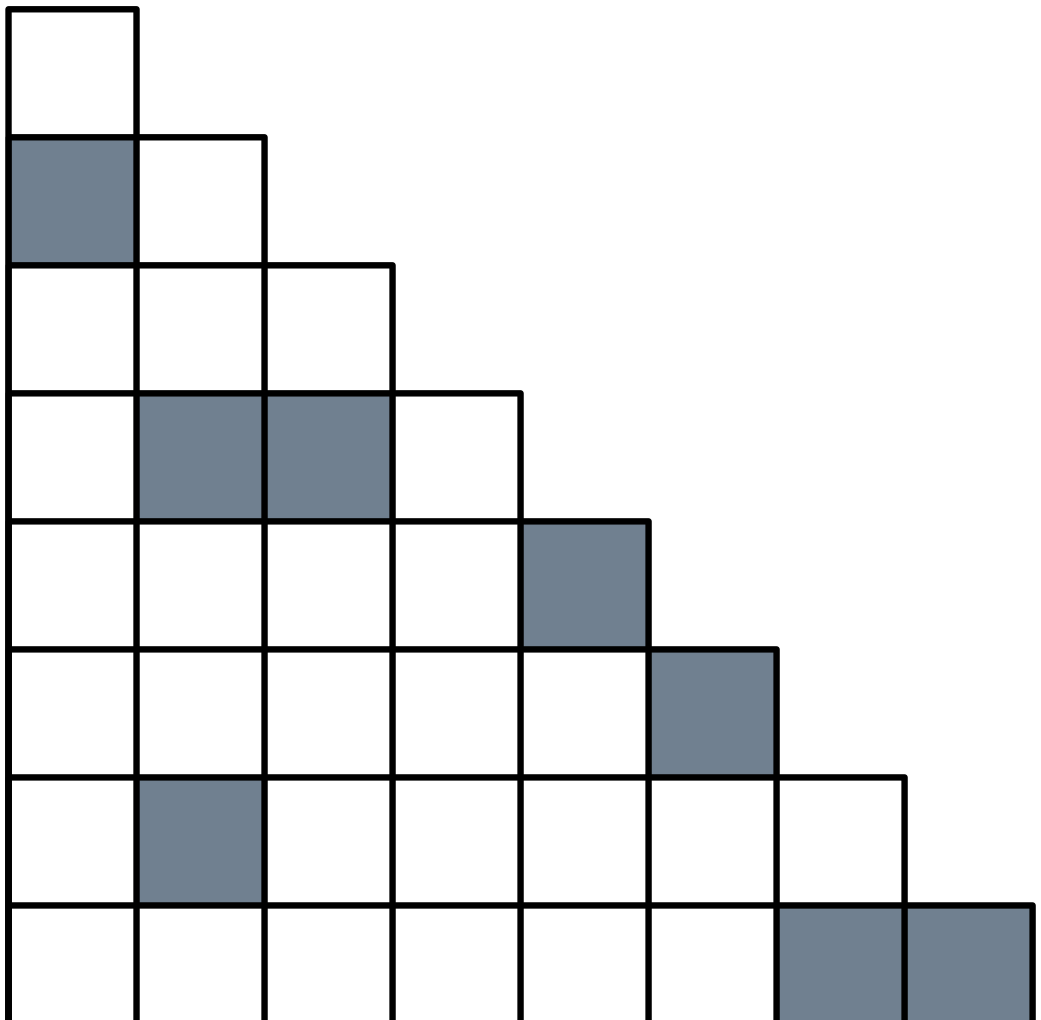
I- The objects

b) Triangle Bases



Filling configurations

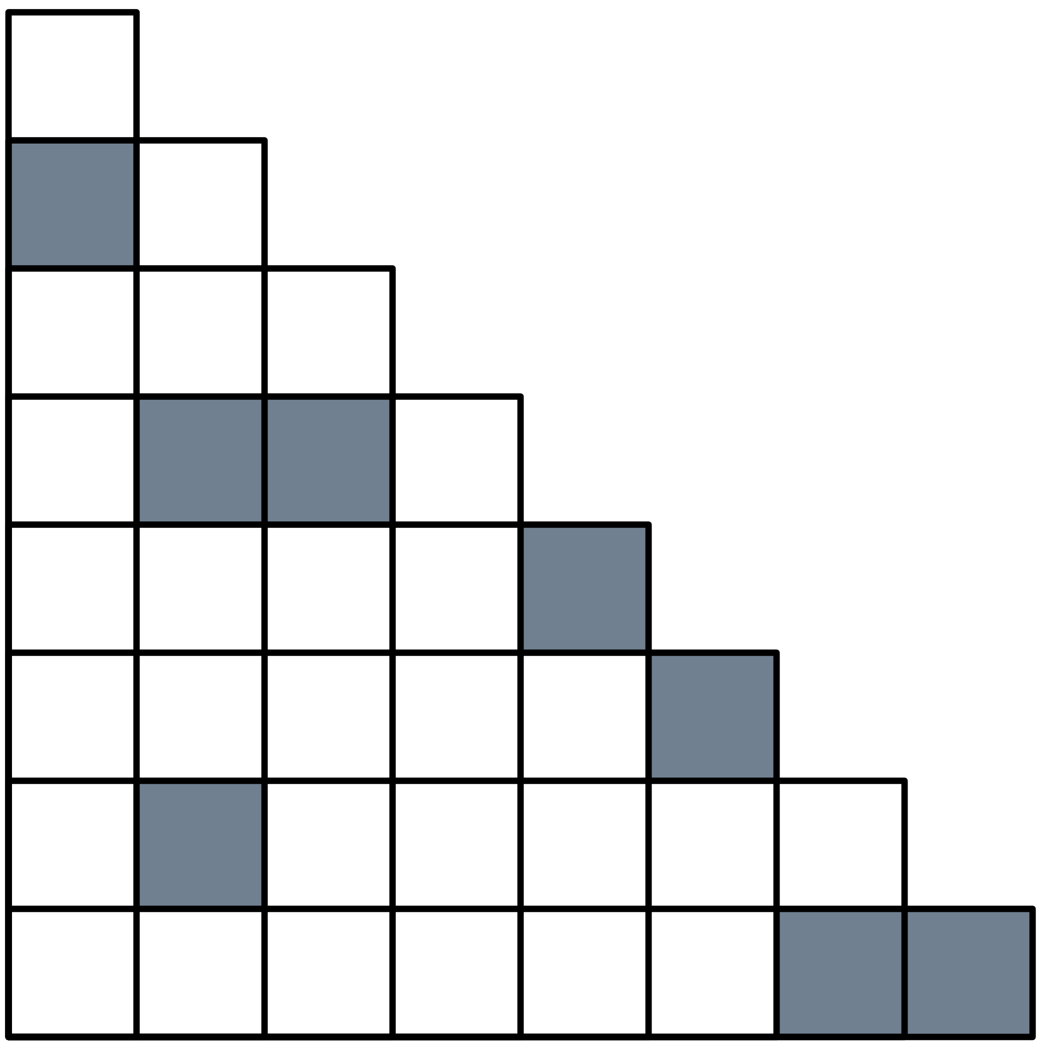
A **configuration** of size n is a set of n cells in the triangle T_n of size n.

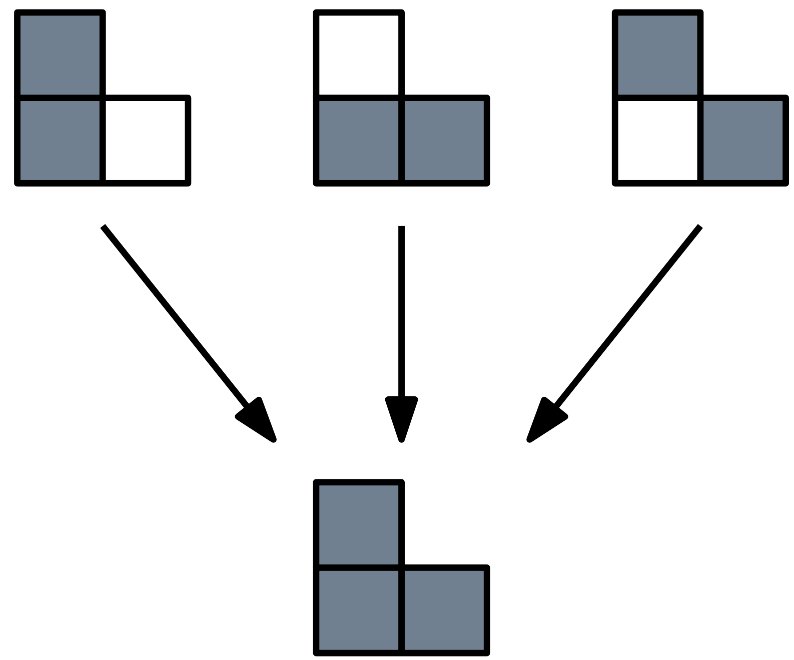


Filling configurations

A **configuration** of size n is a set of n cells in the triangle T_n of size n.

A **filling step** fills the empty cell of a triangle with exactly one empty cell.





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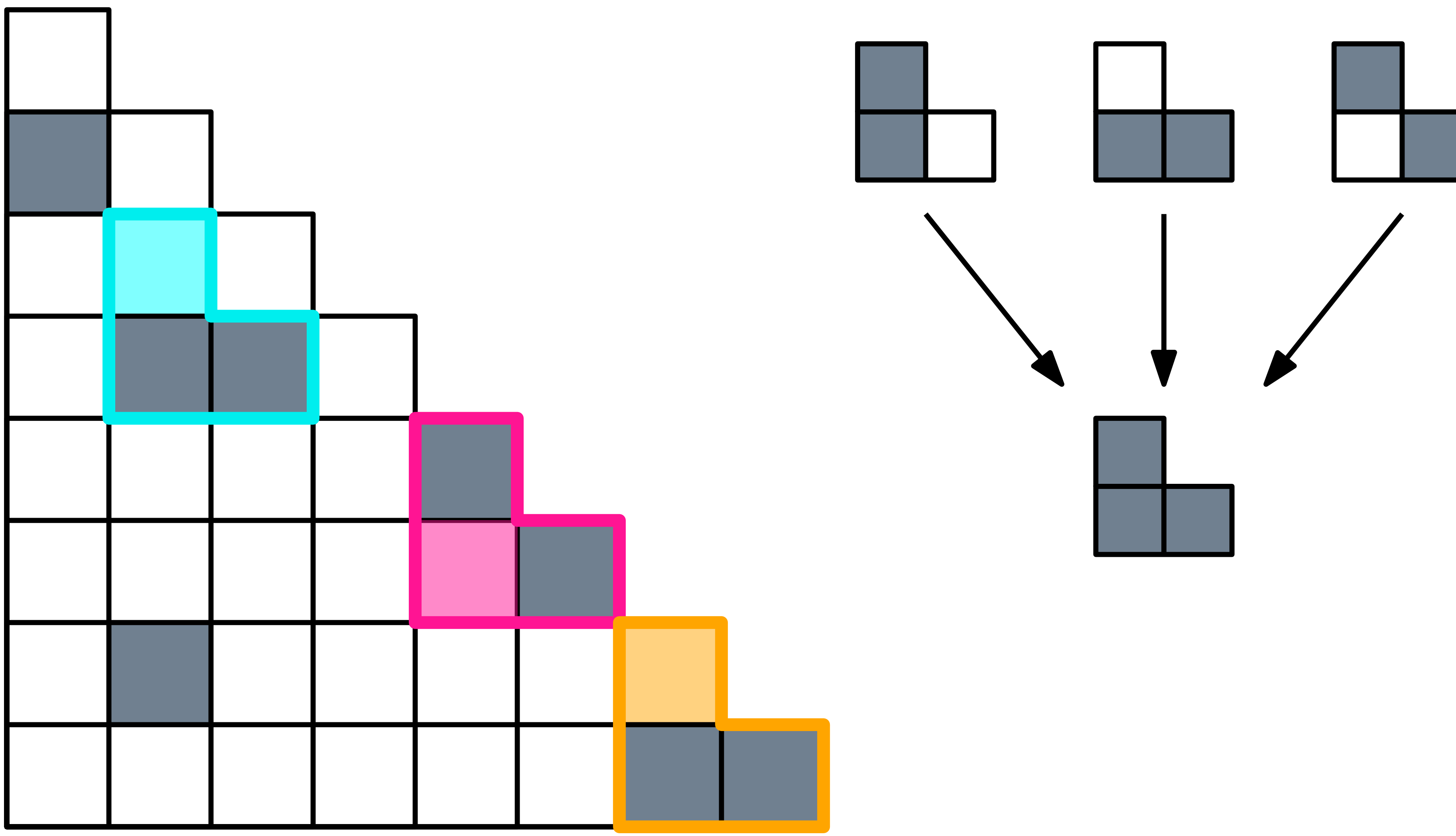

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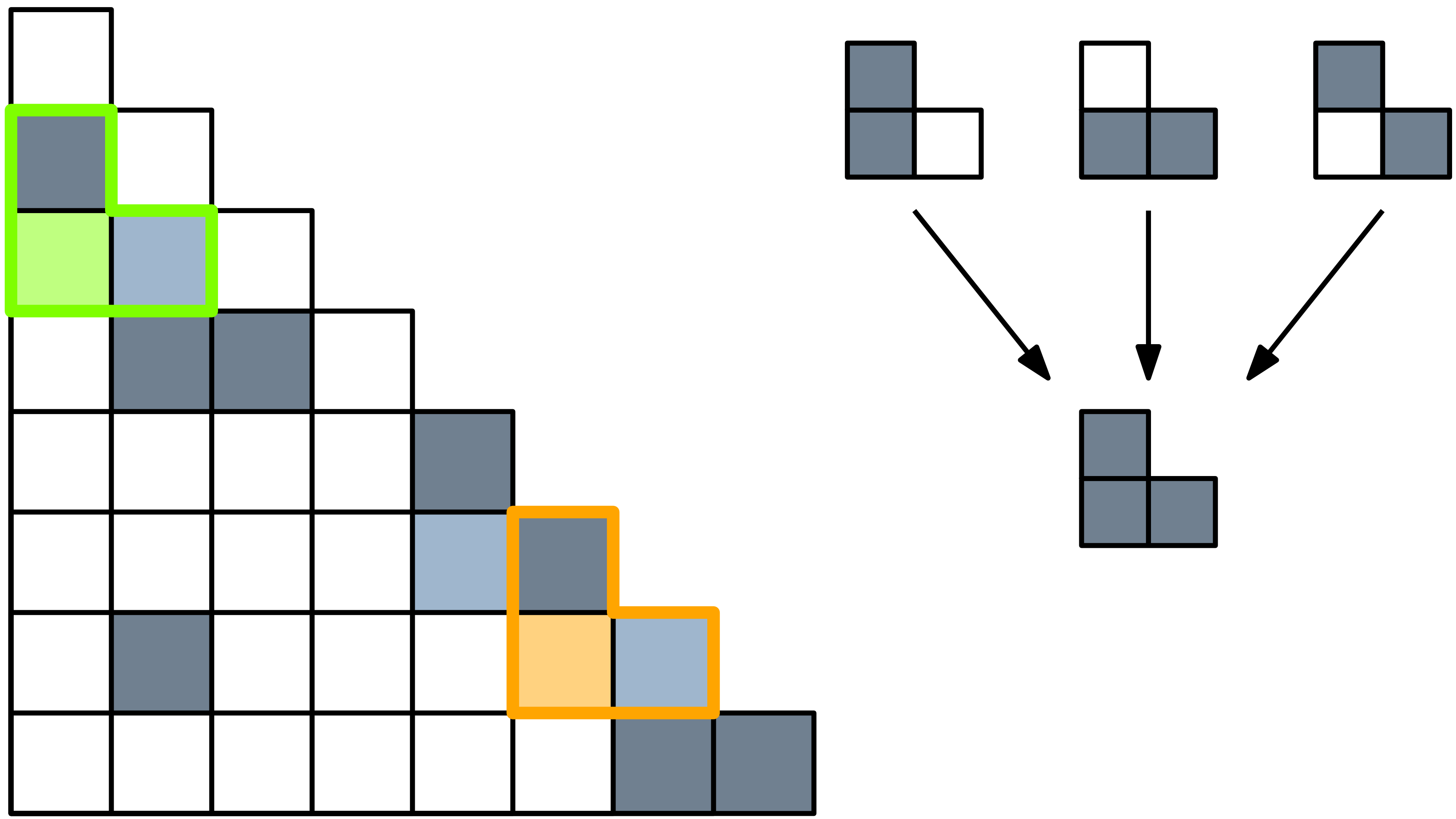

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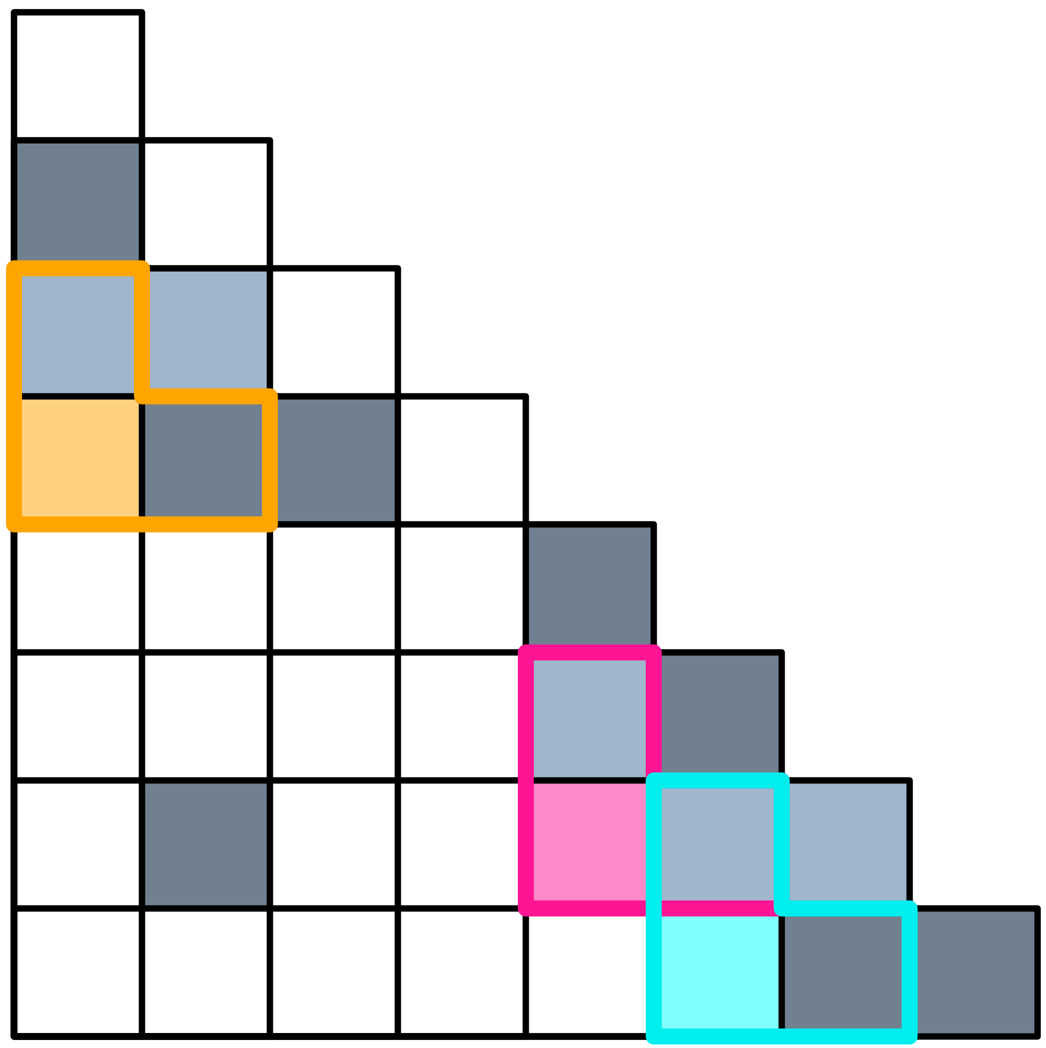

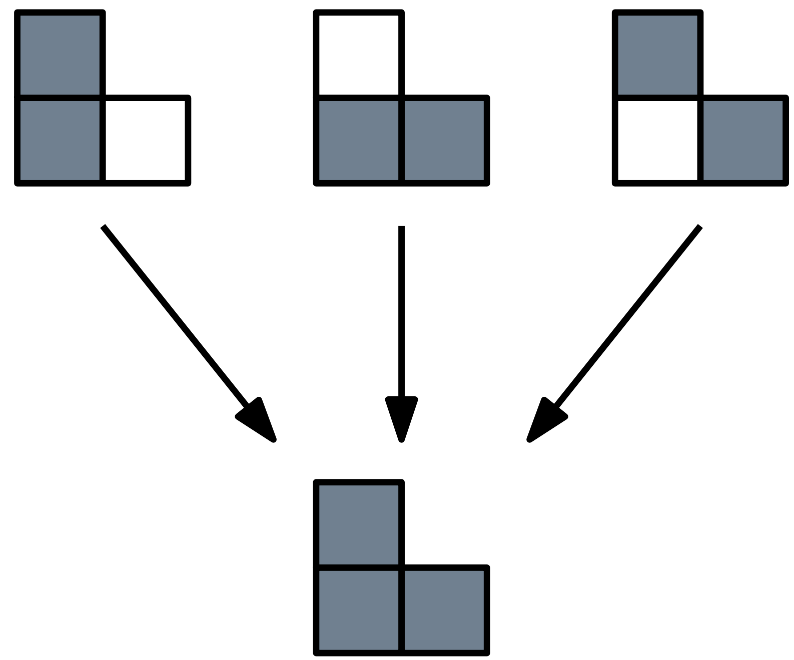

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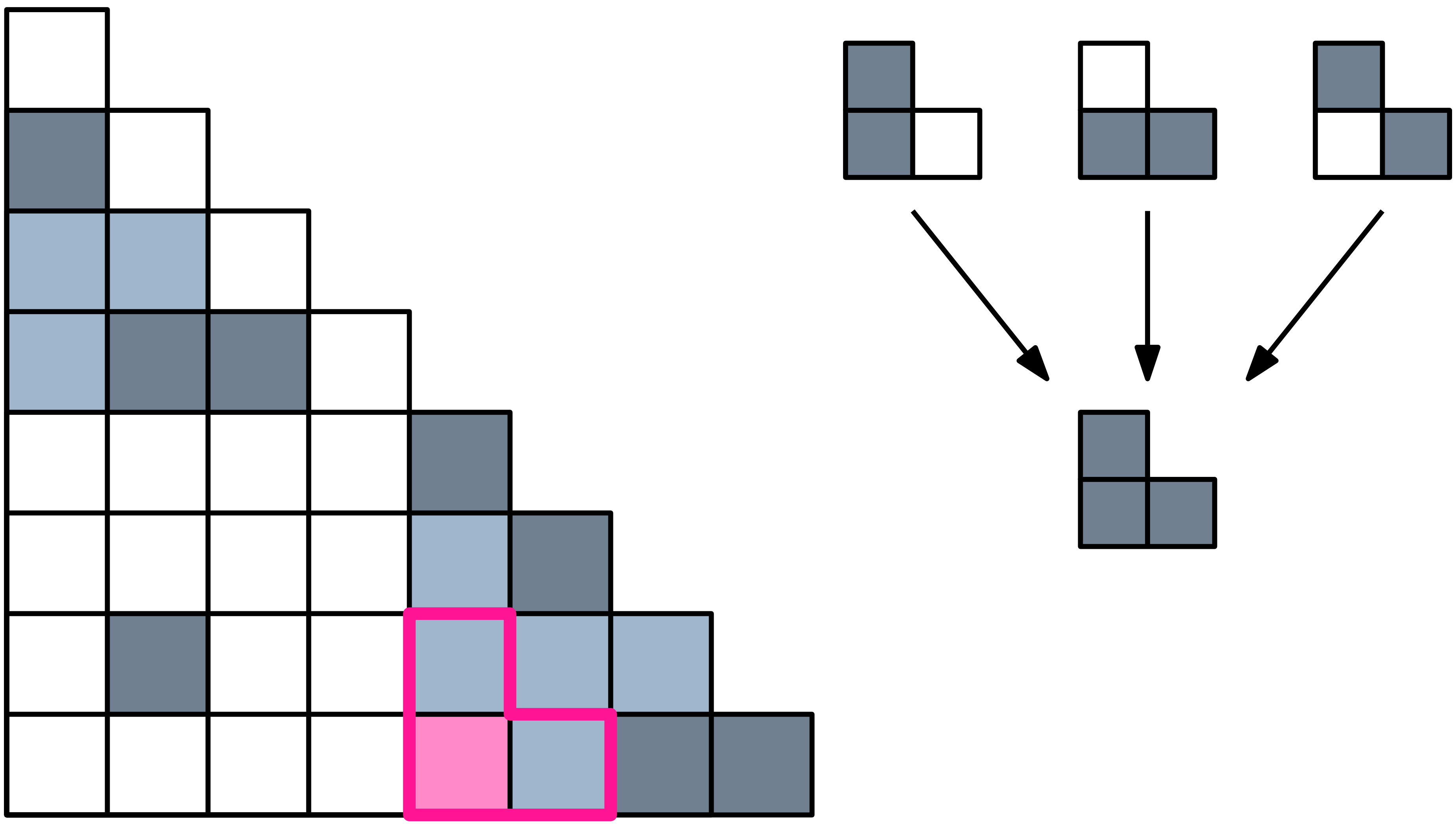

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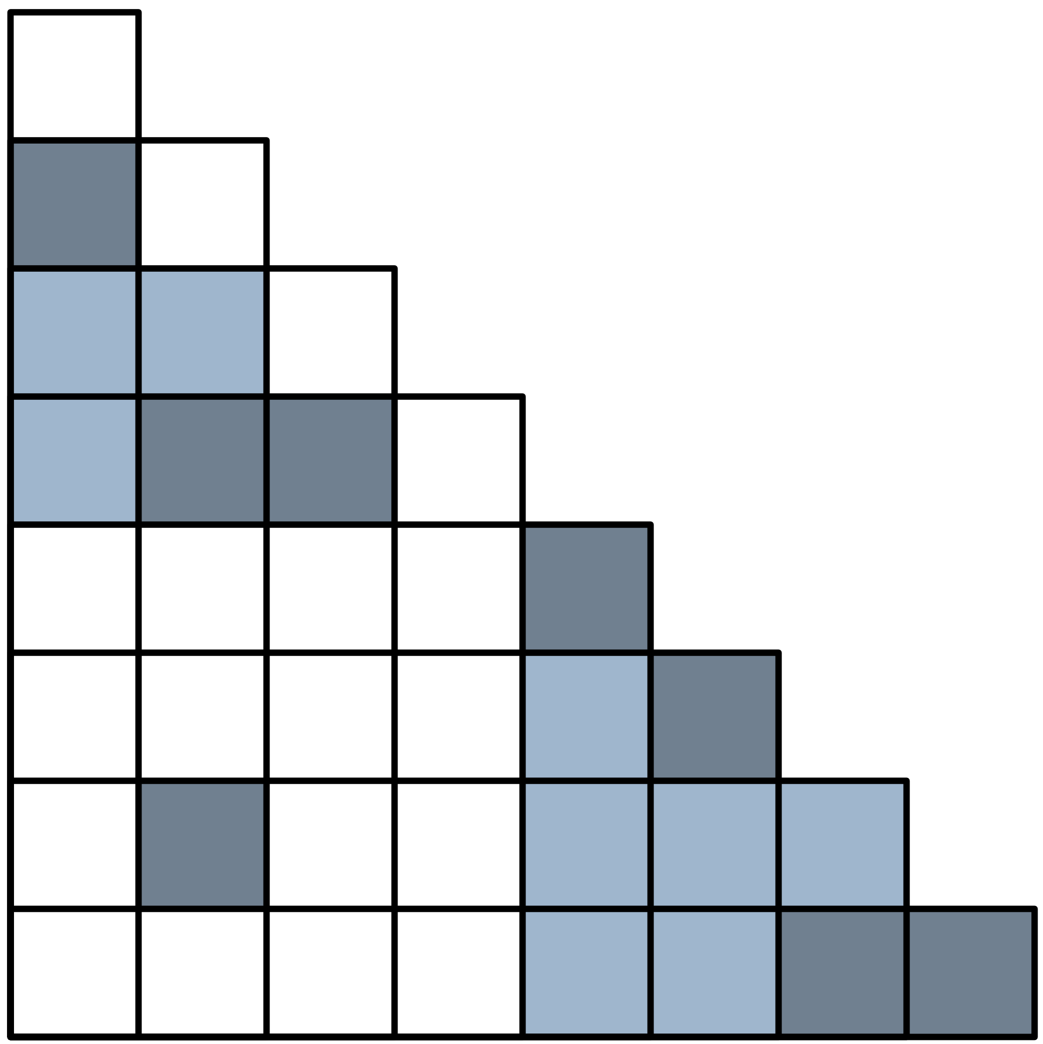
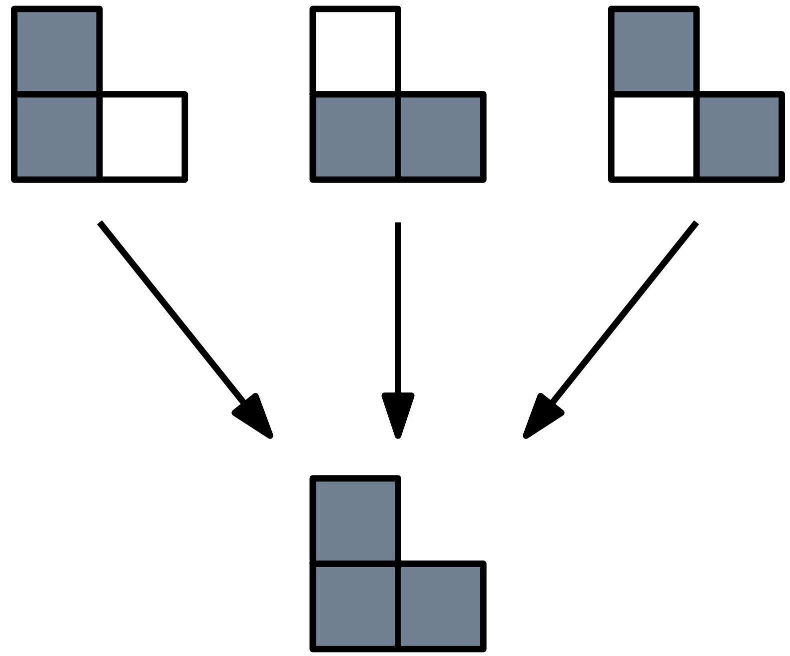
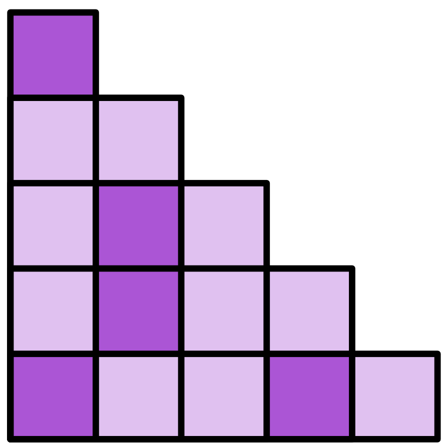


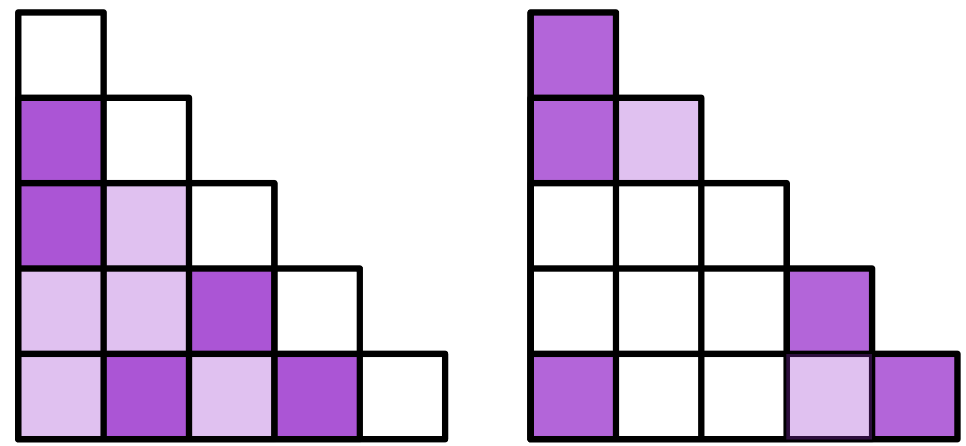


Triangle bases

A **triangle basis** of size n is a configuration of n points that fills T_n. Denote $\mathcal{B}_n$ their set.



A basis.



Not a basis.

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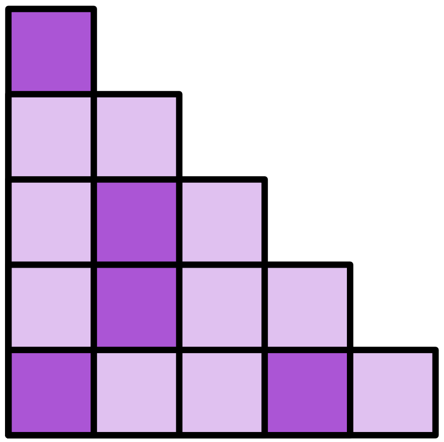
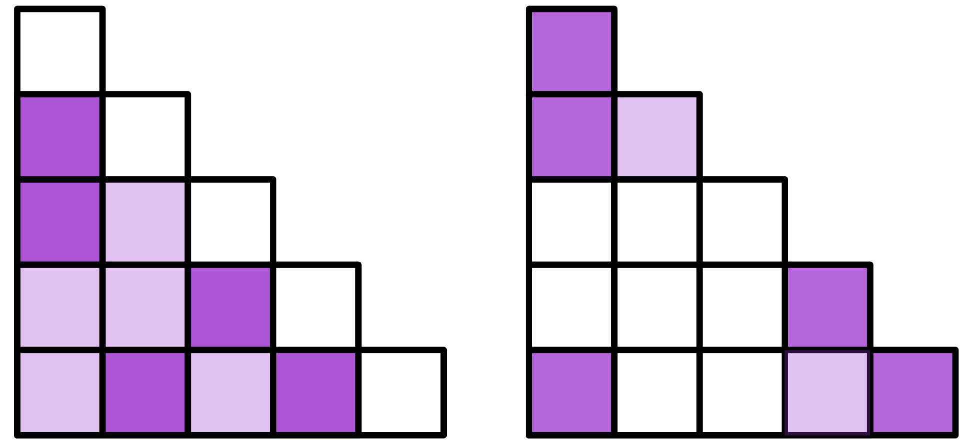


A basis.



Not a basis.

► Used to study "totally extremaly permutive" subshifts, a generalisation of bipermutive cellular automata [Salo '20].

Triangle bases

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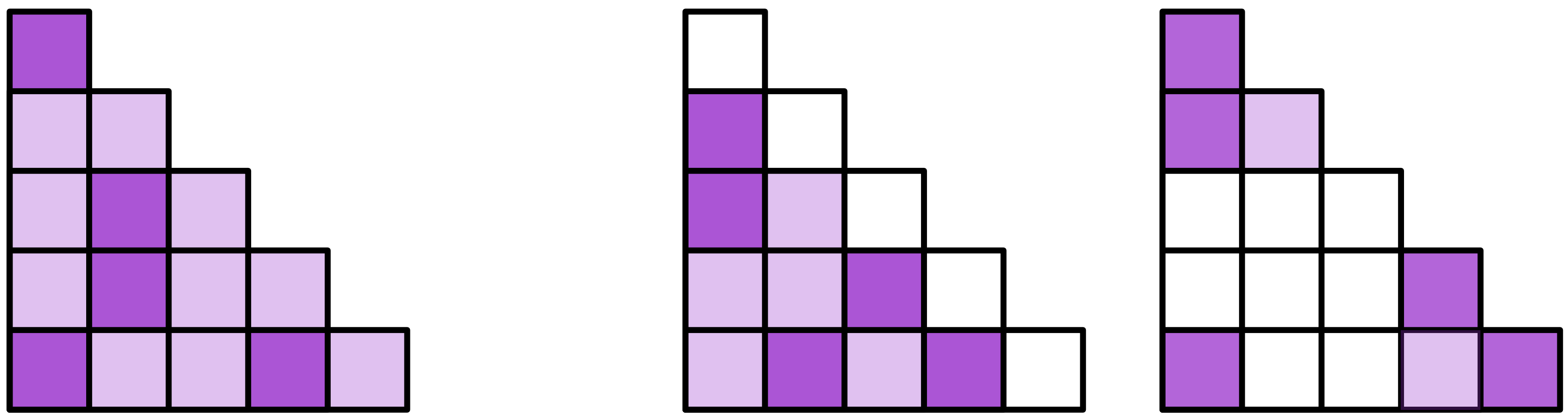
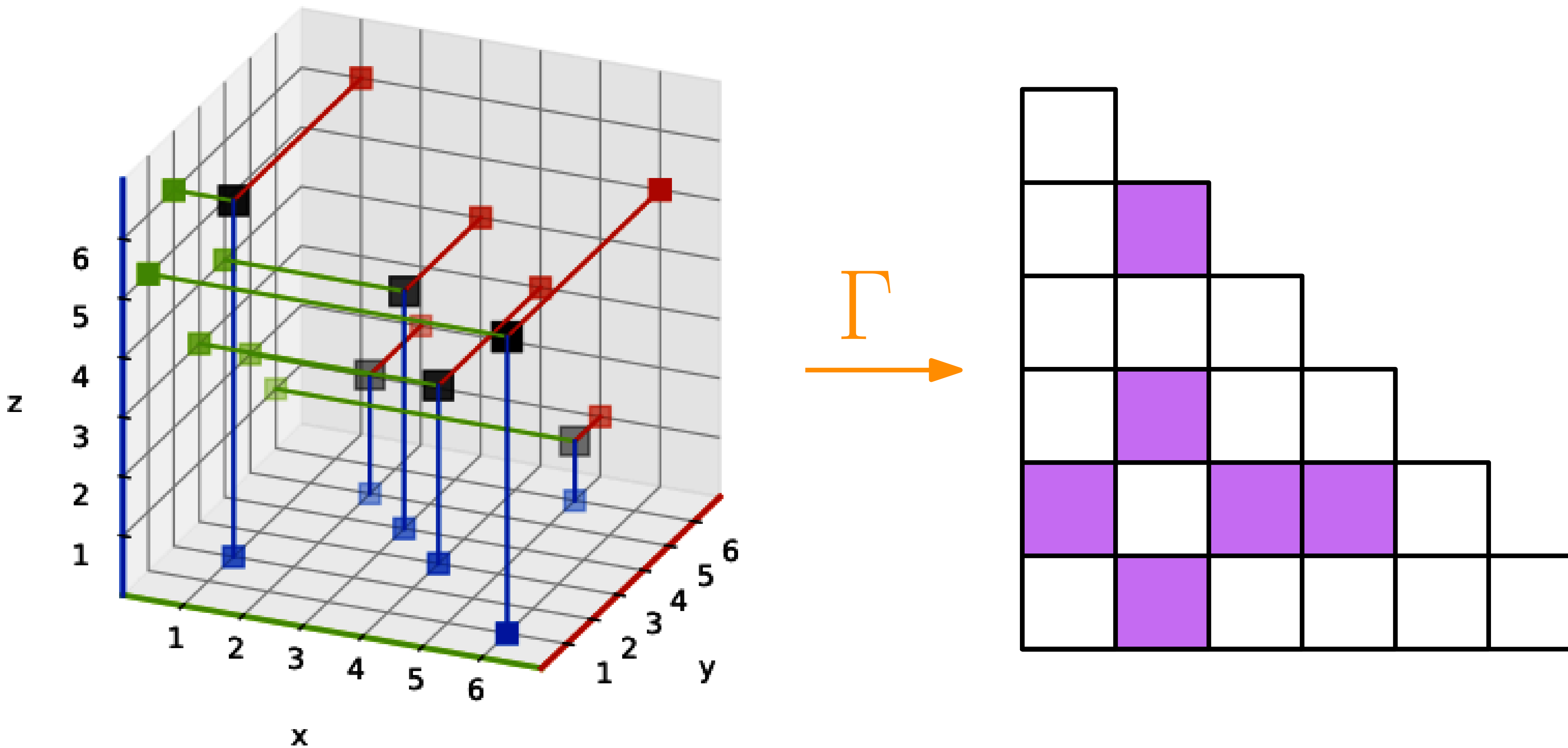


A basis. Not a basis.

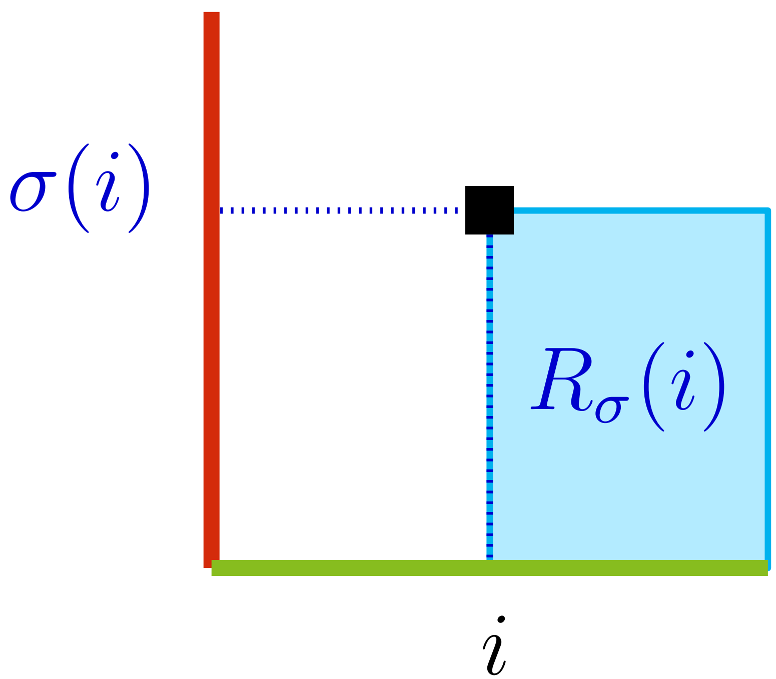
Theorem. [S. '25] For all n, the set of triangle bases of size n is in bijection with $Av_n((12, 12), (312, 231))$.

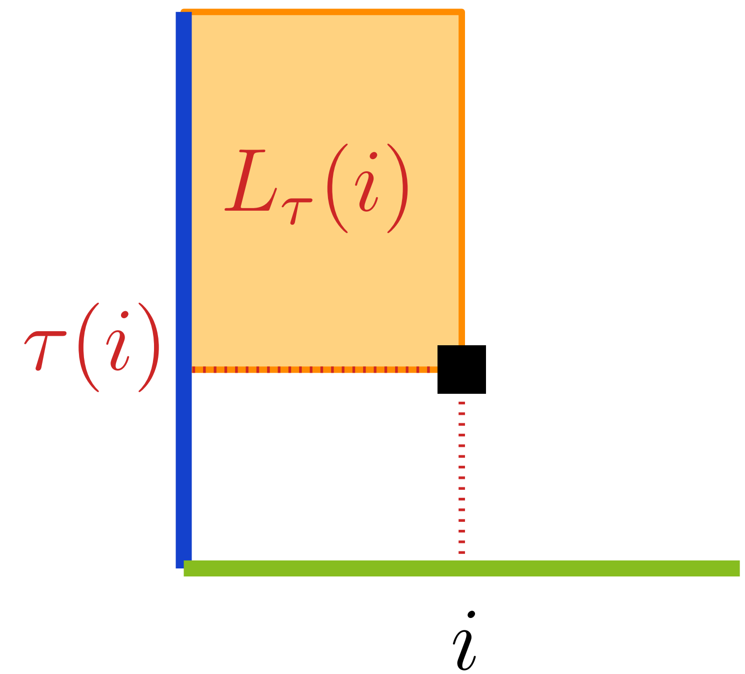
II- A bijection



The bijection

An **inversion** of $\sigma \in \mathfrak{S}_n$ is $(i, j) \in [\![1, n]\!]$ with $i < j$ and $\sigma(i) > \sigma(j)$.





Right inversion set at i
$r_\sigma(i) = |R_\sigma(i)|$

Left inversion set at i
$\ell_\tau(i) = |L_\tau(i)|$

The bijection from 3-permutations to bases:

$$\Gamma : (\sigma, \tau) \mapsto \{(r_\sigma(i), \ell_\tau(i)) \mid i \in [\![1, n]\!]\}$$

The bijection

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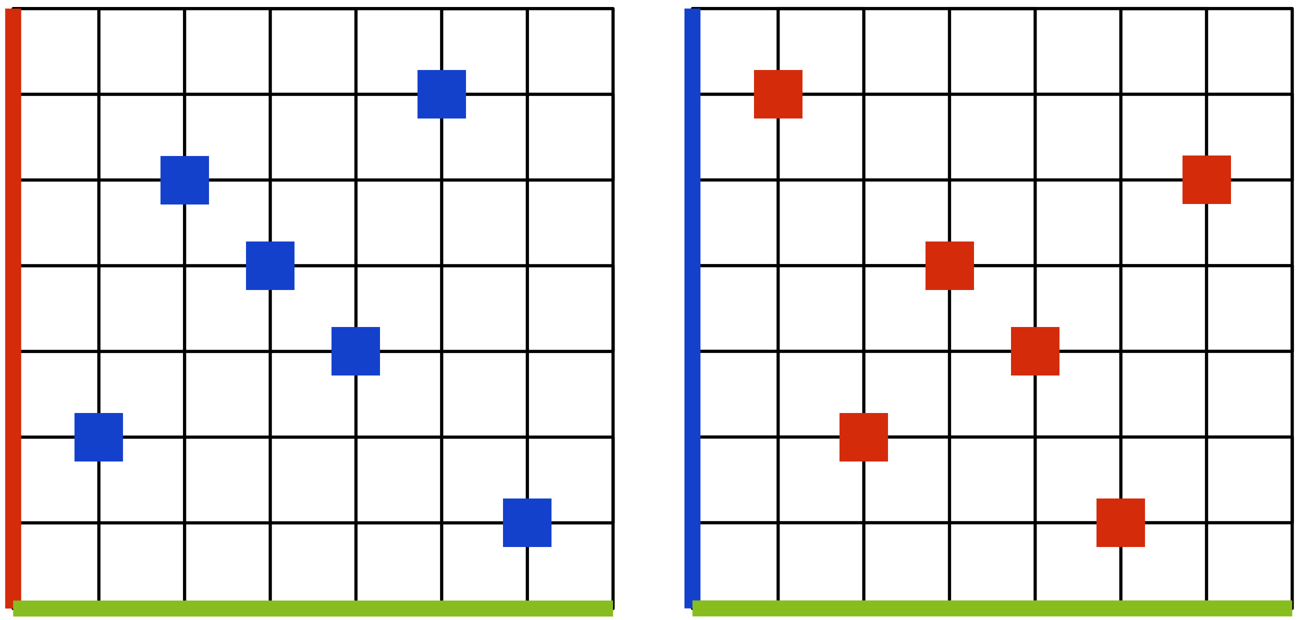
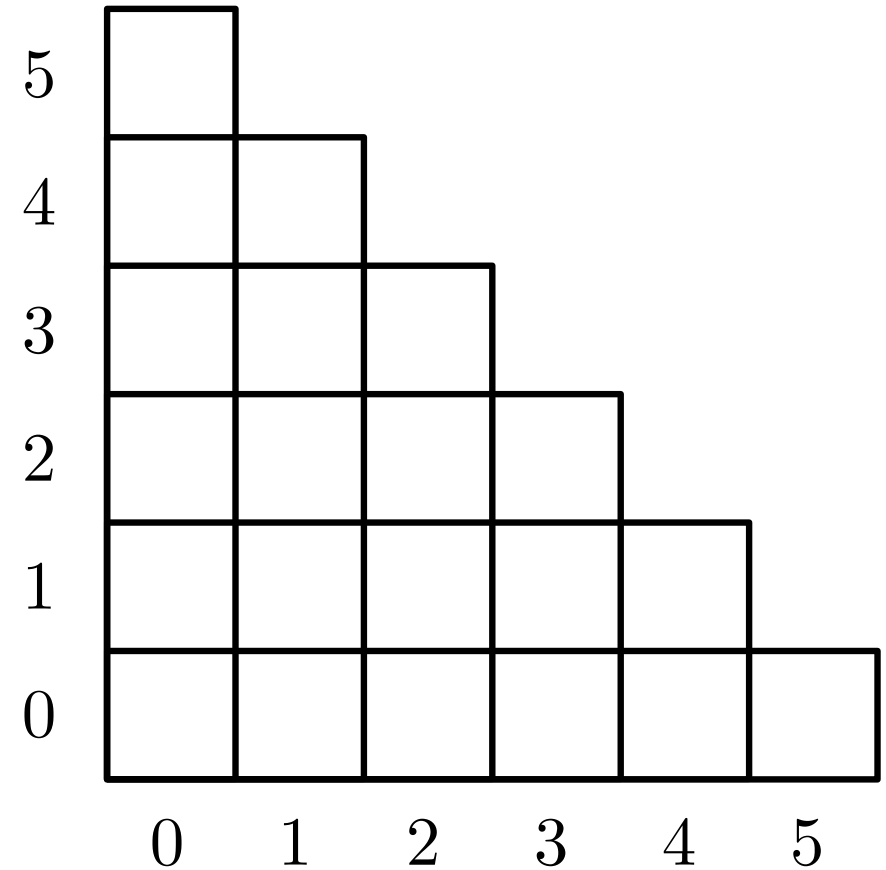
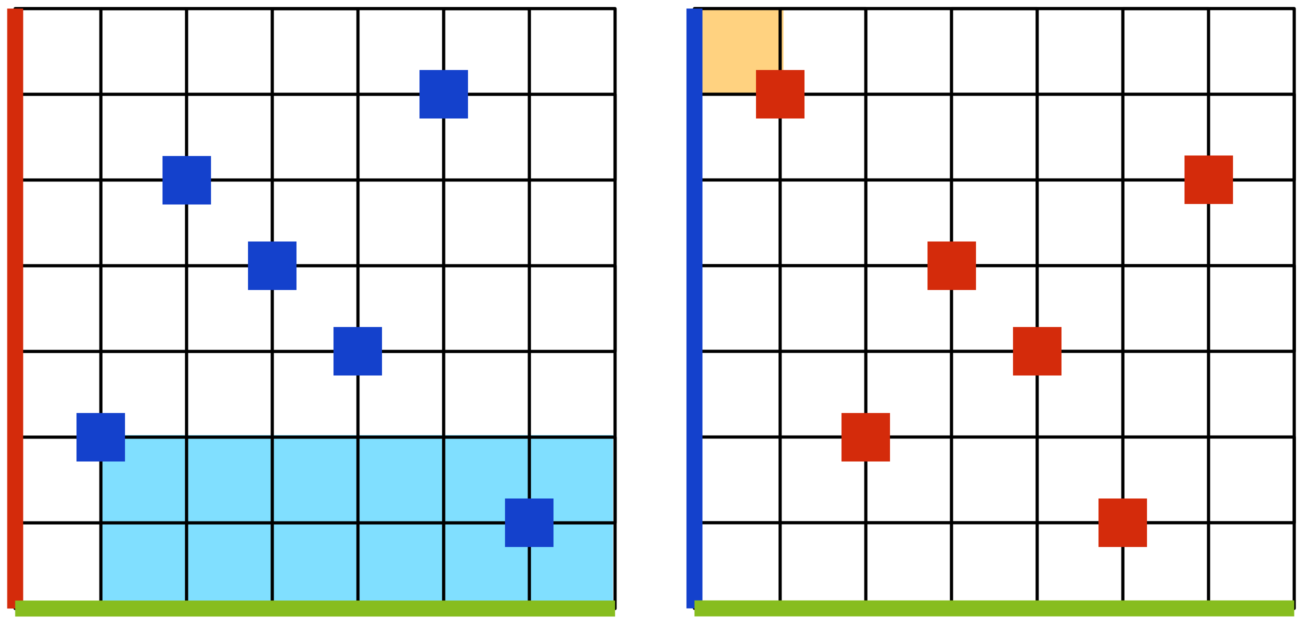


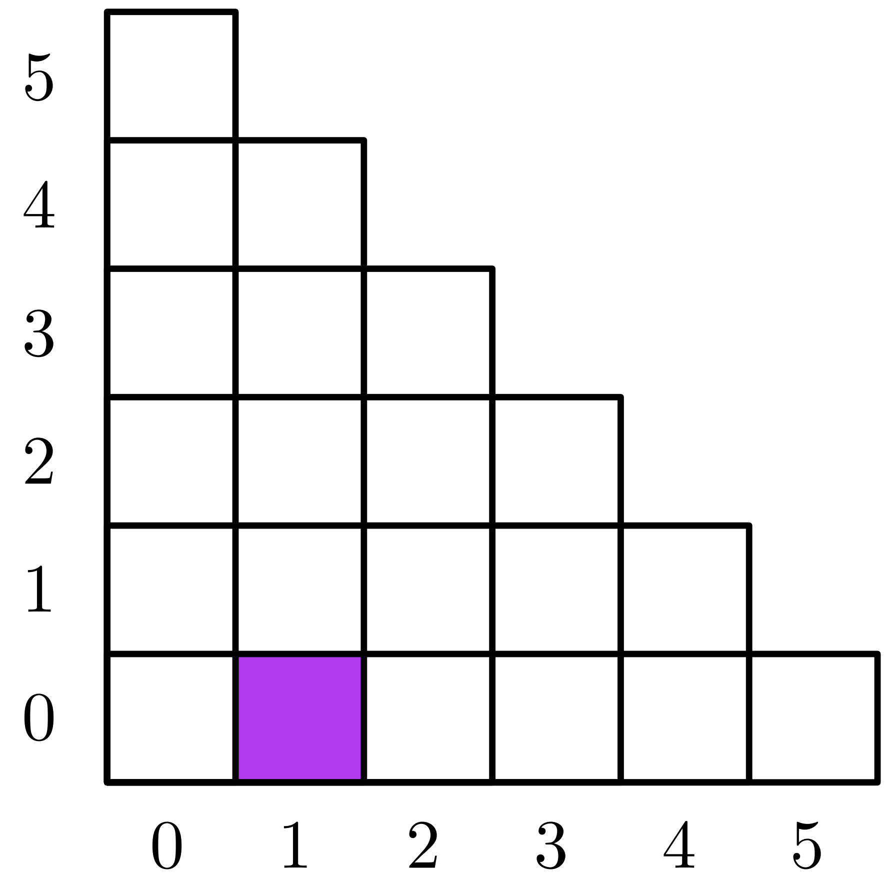


The bijection

$$\Gamma : (\sigma, \tau) \mapsto \{(r_\sigma(i), \ell_\tau(i)) \mid i \in [\![1, n]\!]\}$$



$$i = 1 \mapsto (1, 0)$$



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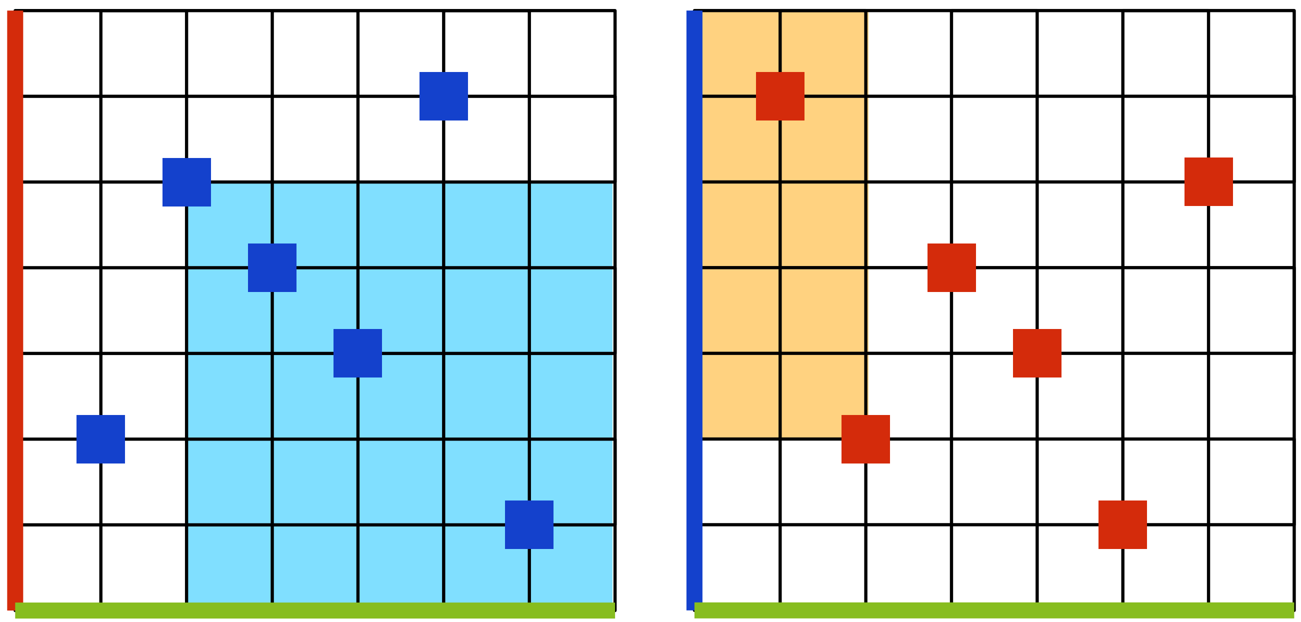
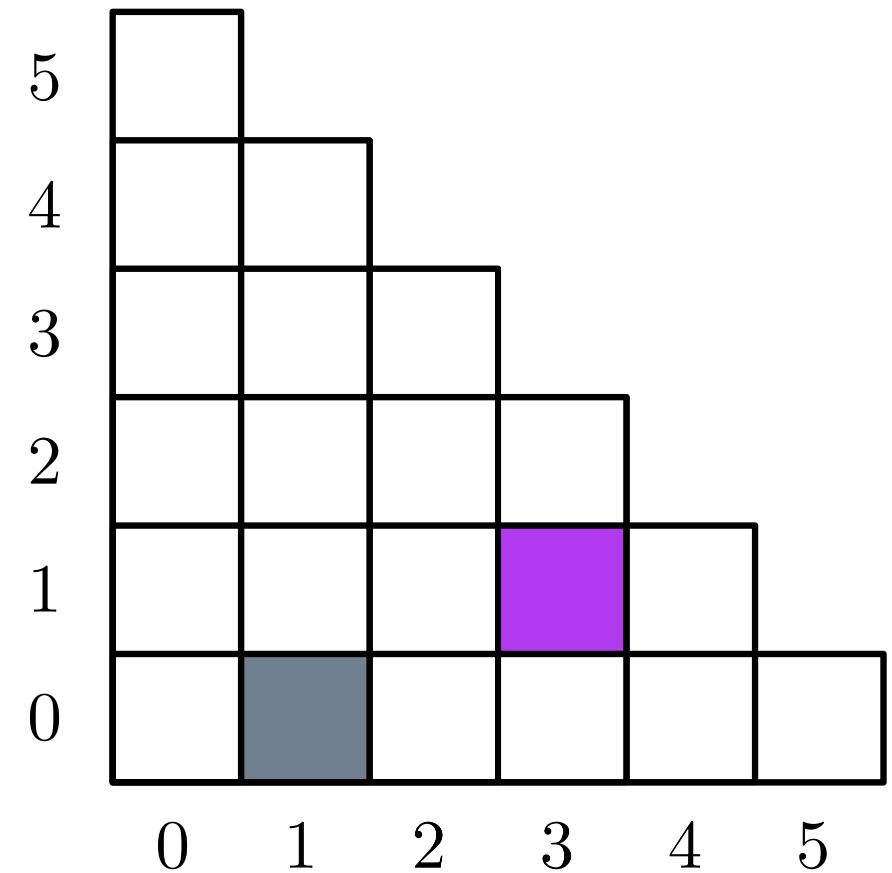


$$i = 2 \mapsto (3, 1)$$



The bijection

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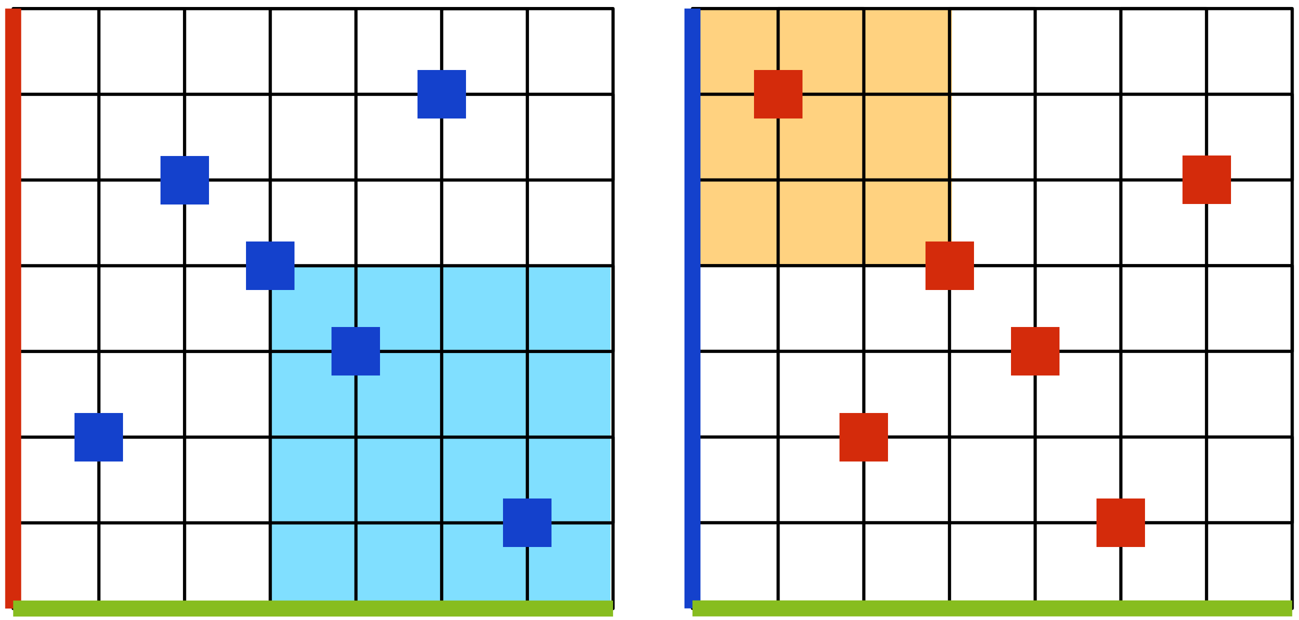
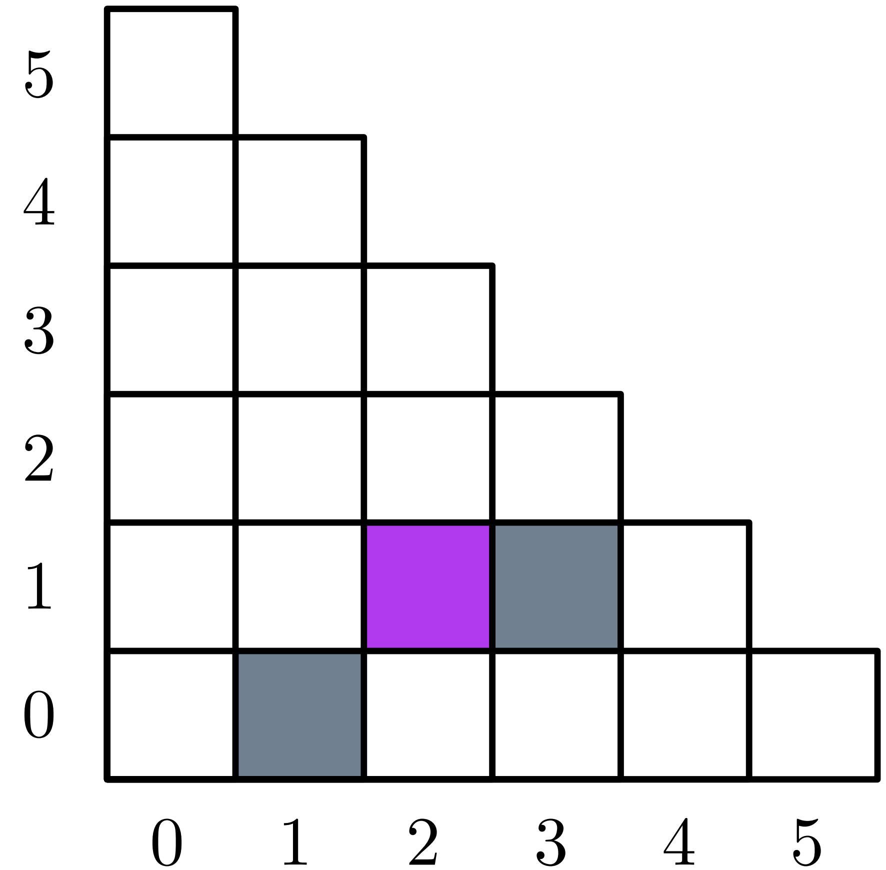
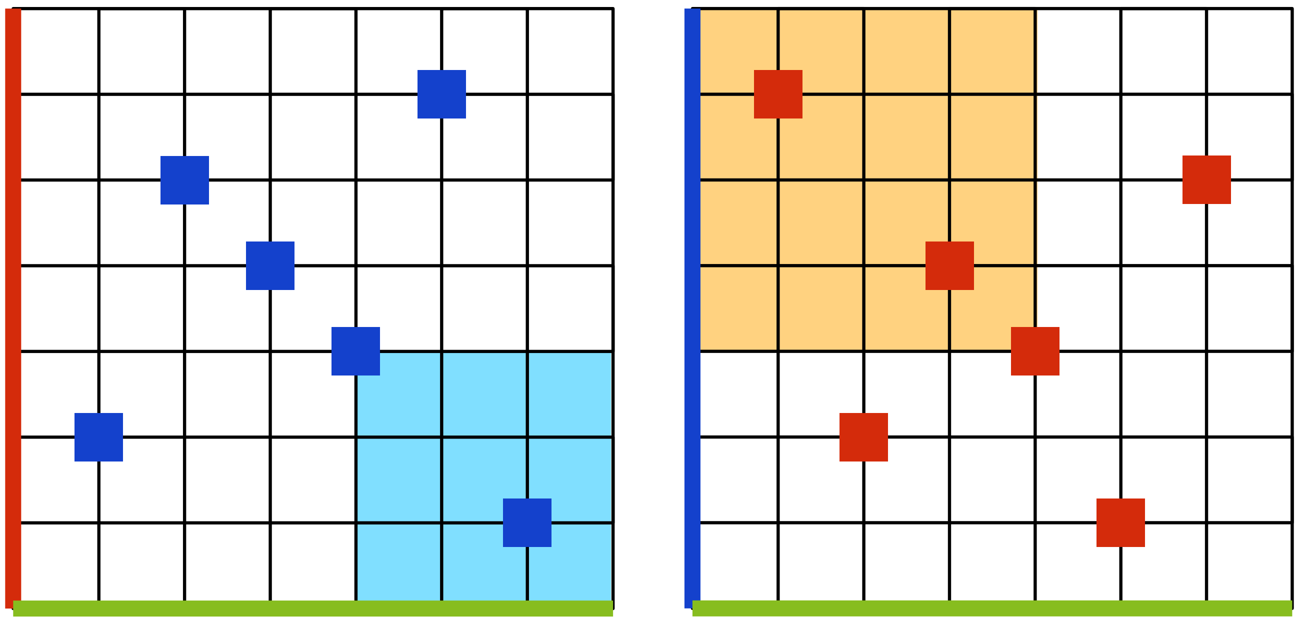


$$i = 3 \mapsto (2, 1)$$

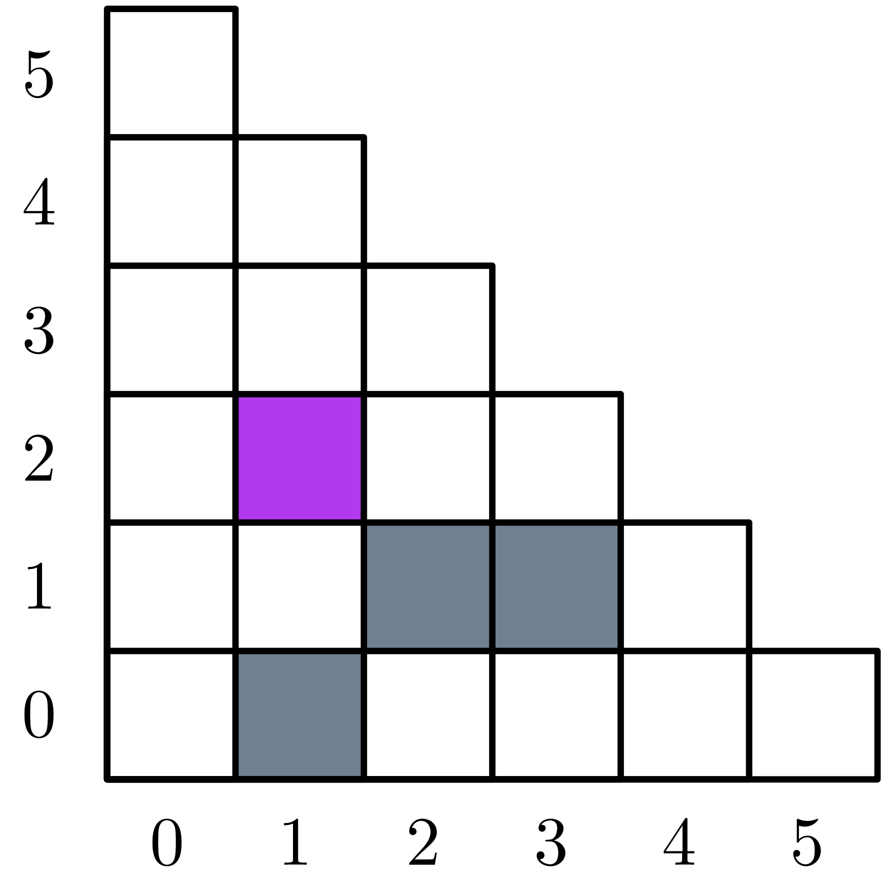


The bijection

$$\Gamma : (\sigma, \tau) \mapsto \{(r_\sigma(i), \ell_\tau(i)) \mid i \in [\![1, n]\!]\}$$



$$i = 4 \mapsto (1, 2)$$



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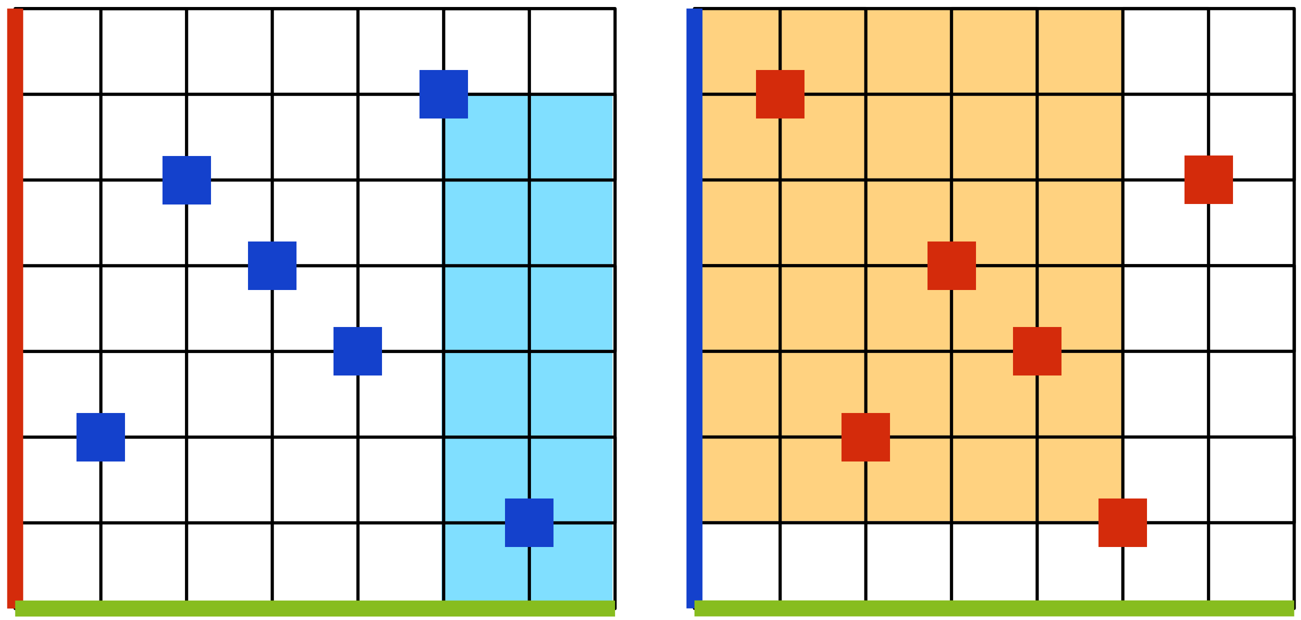
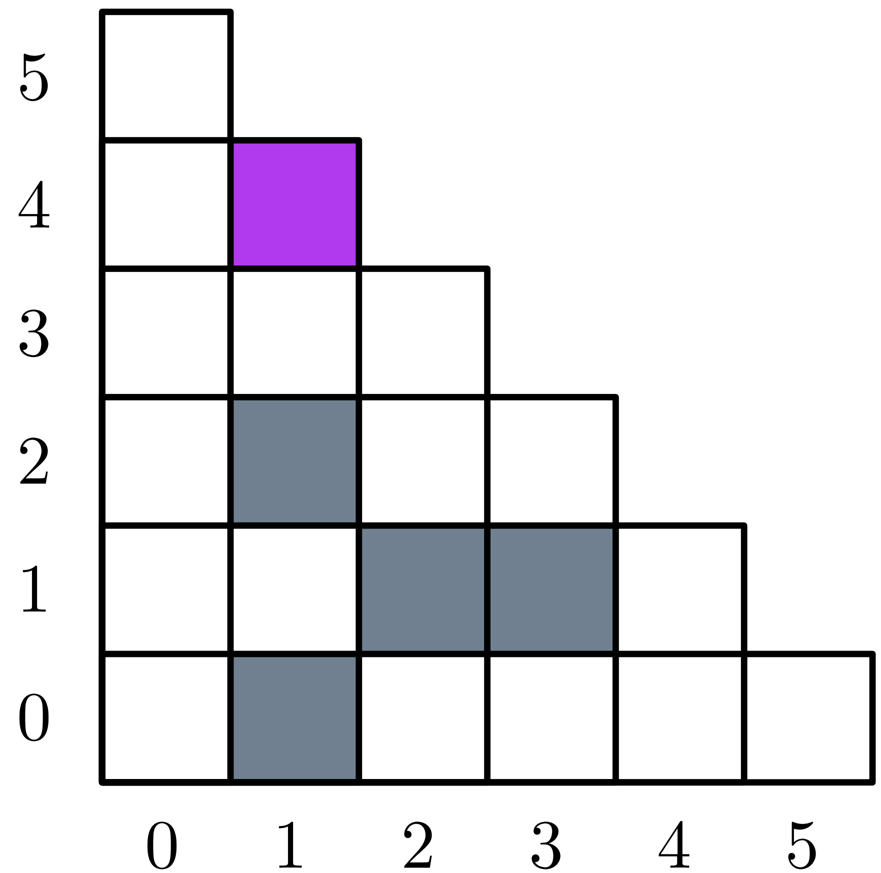


$$i = 5 \mapsto (1, 4)$$



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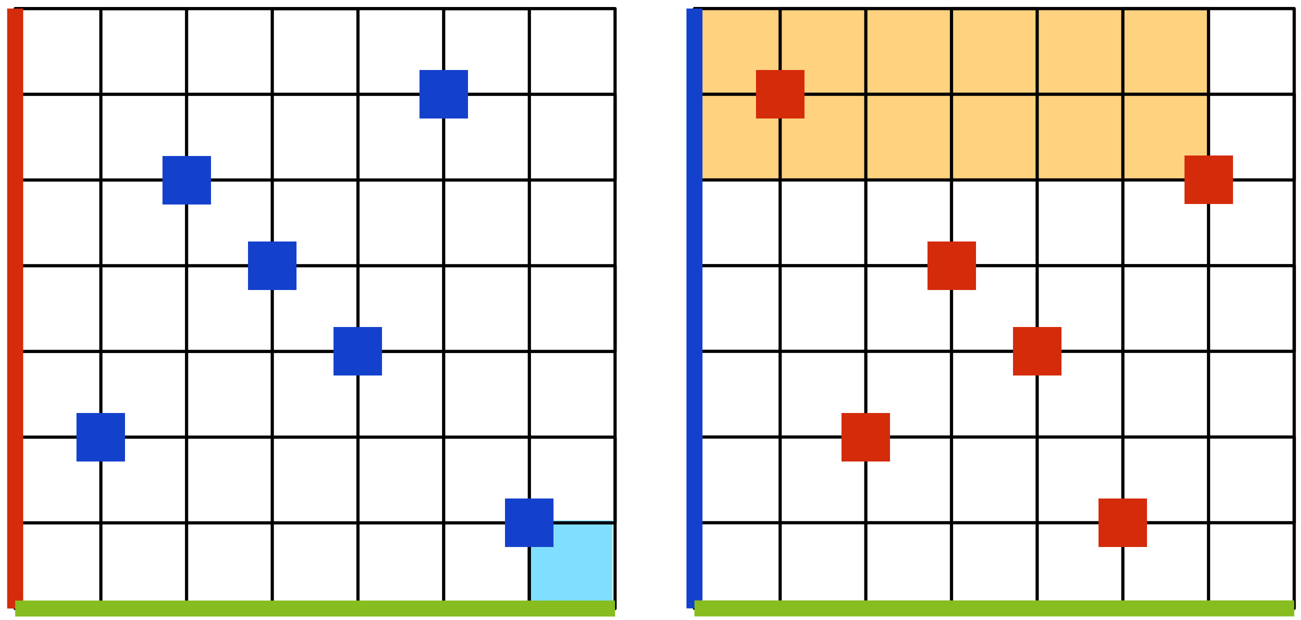
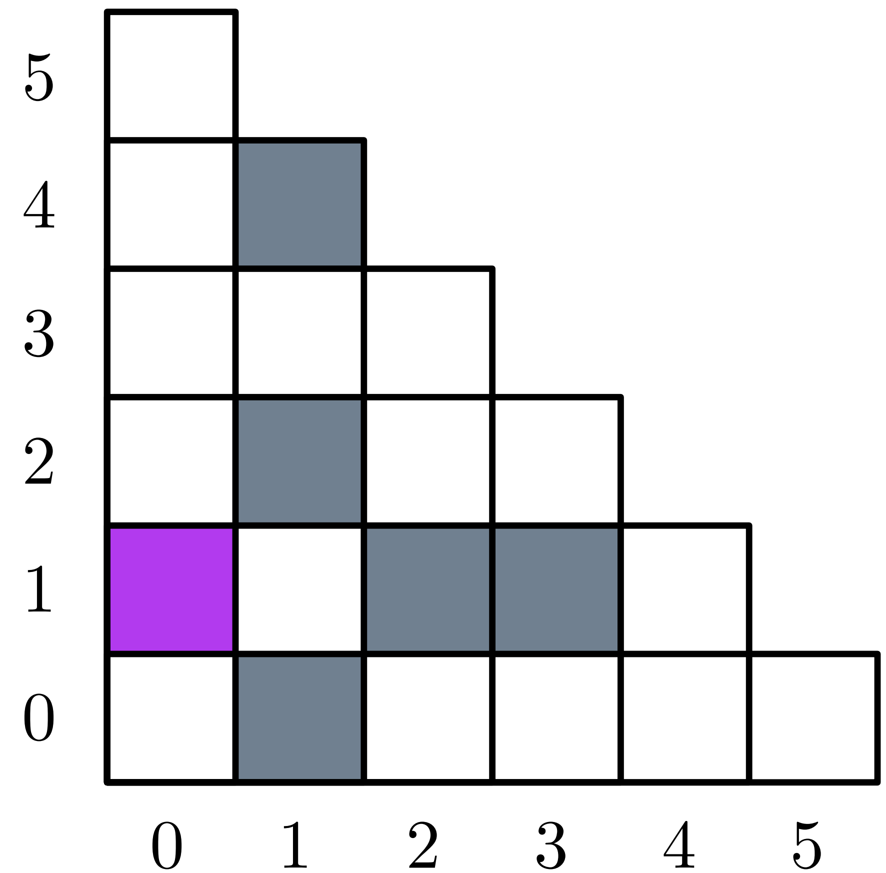


$$i = 6 \mapsto (0, 1)$$



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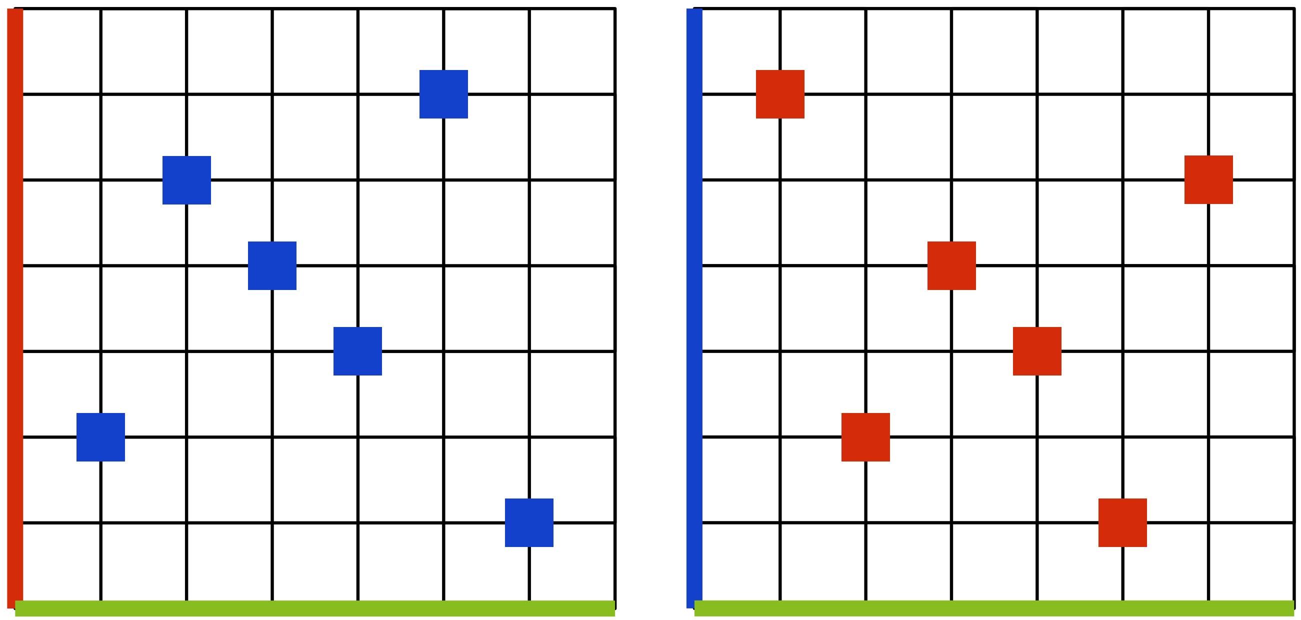
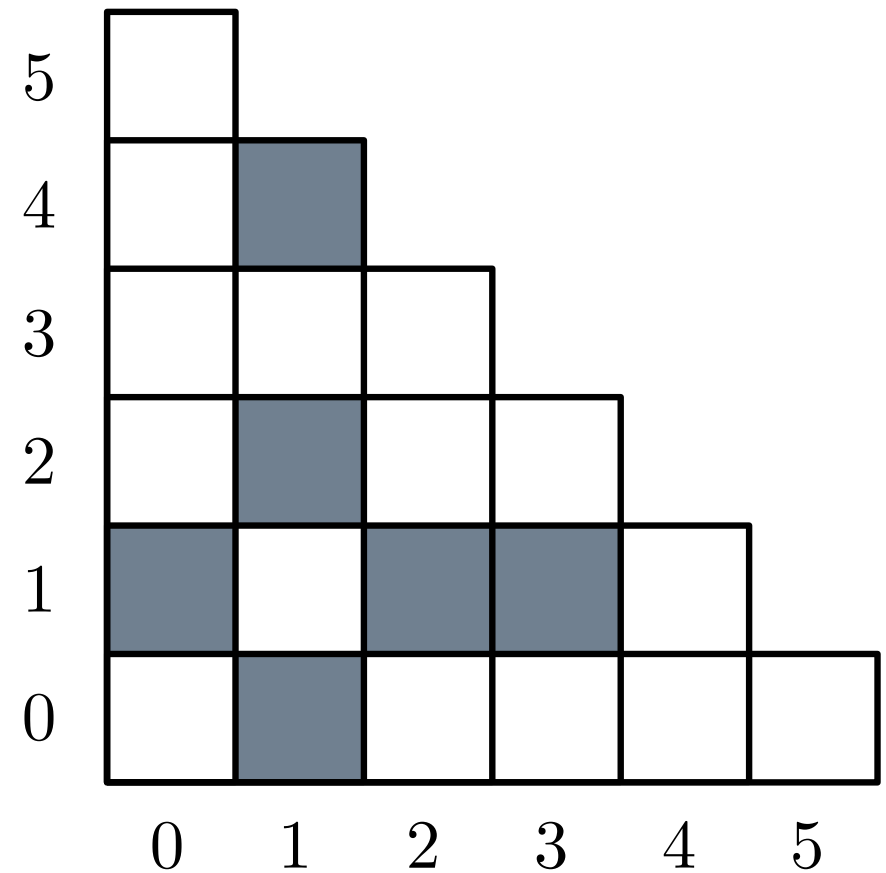
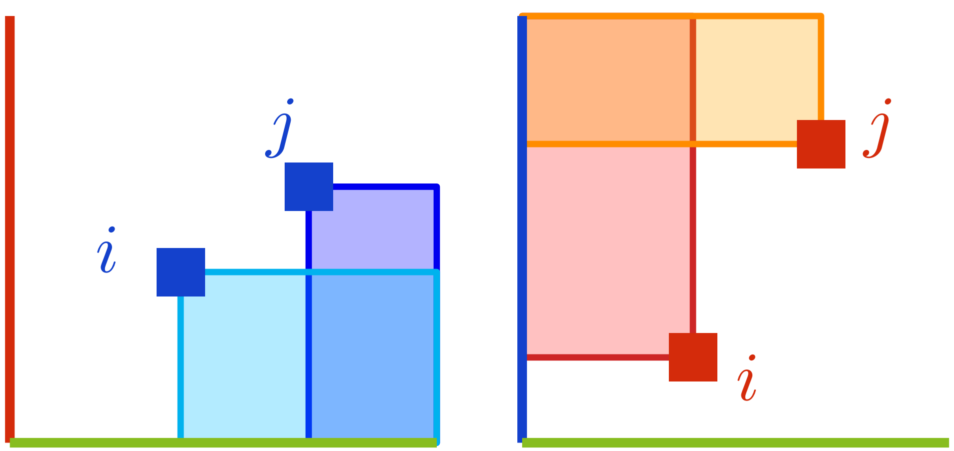




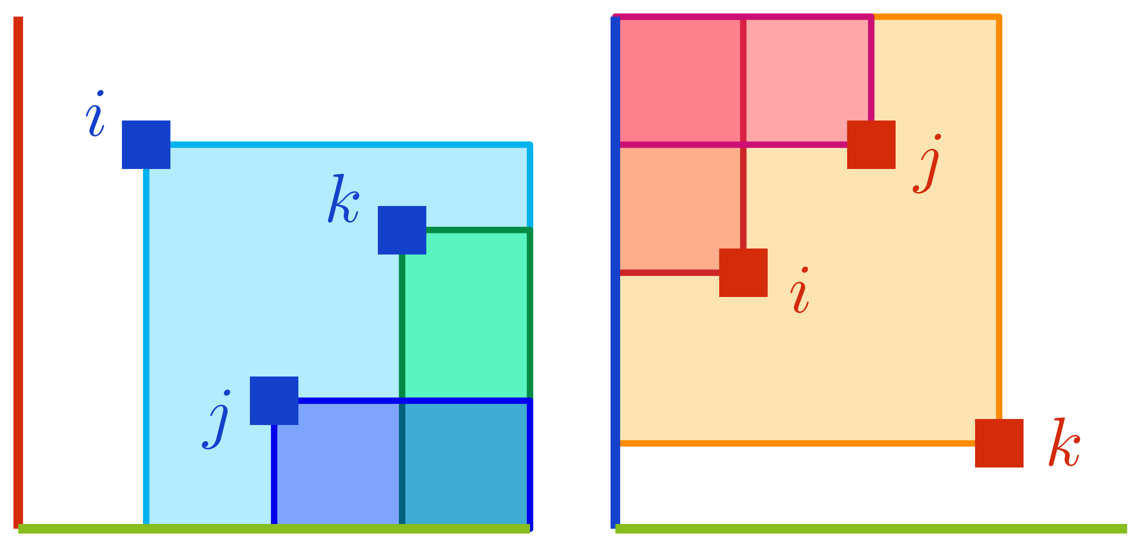
Theorem. [S. '25] For all n, Γ is a bijection between $Av_n((12, 12), (312, 231))$ and the triangle bases of size n.

Why does avoiding $(12, 12)$ and $(312, 231)$ lead to a triangle basis?

Intuition

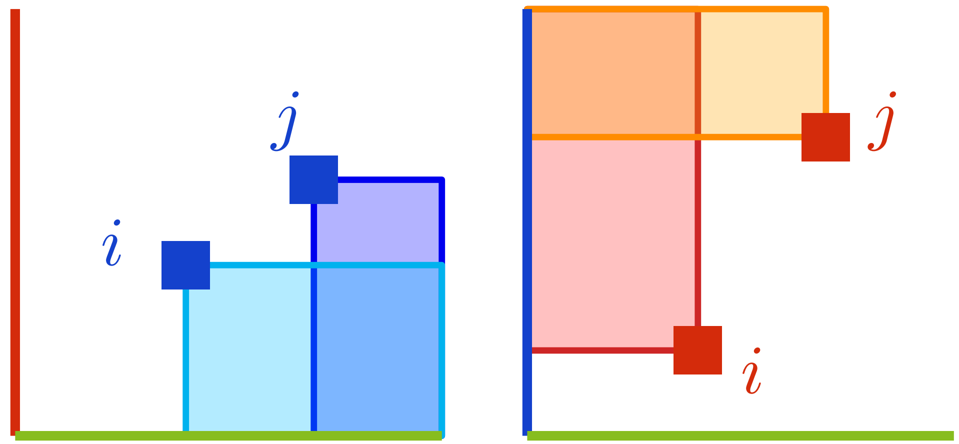


$(12, 12)$



$(312, 231)$

Avoiding $(12, 12)$: no "points too close"



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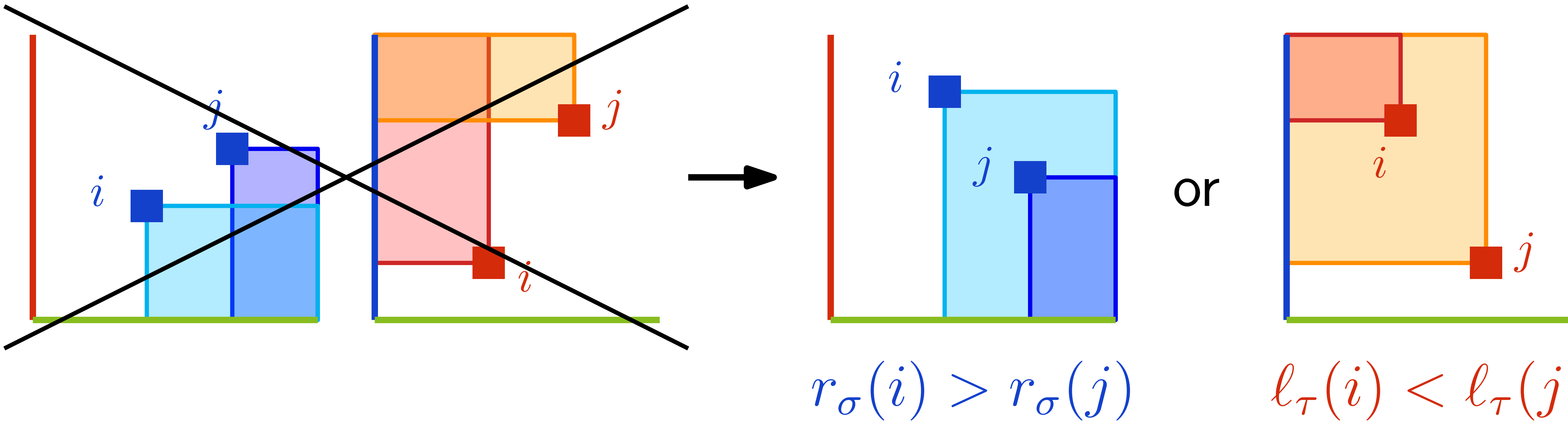
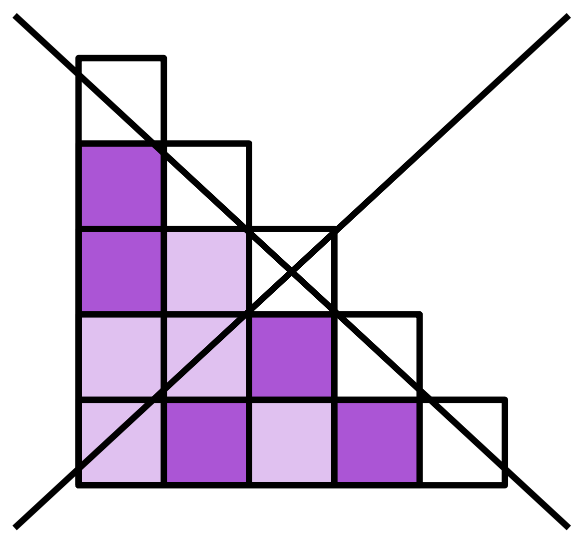


Consequence: If (σ, τ) avoids $(12, 12)$ then

- all points $(r_\sigma(i), \ell_\tau(i))$ are distinct
- the configuration is **sparse**: there is no triangle T of size k such that $|C \cap T| > k$.



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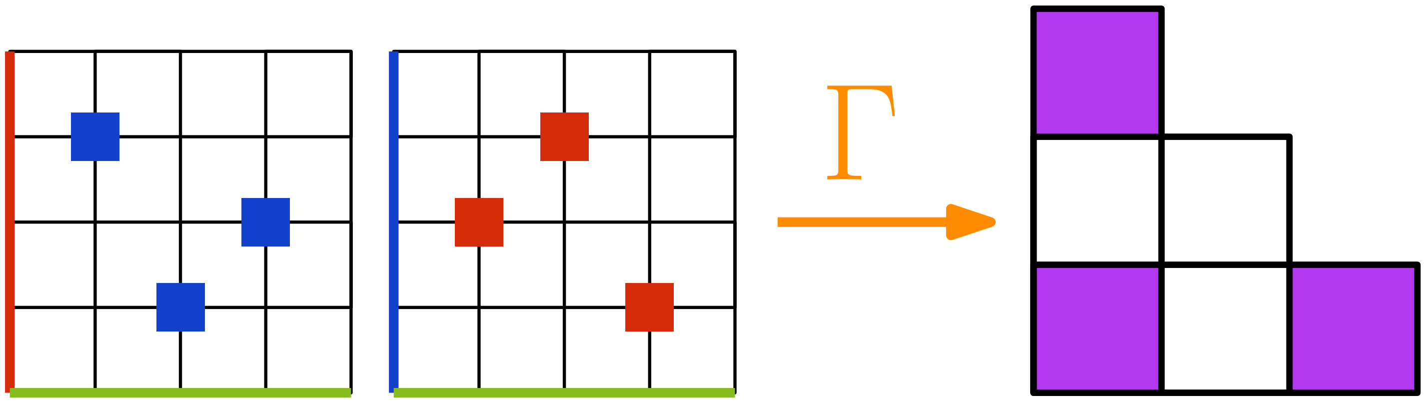
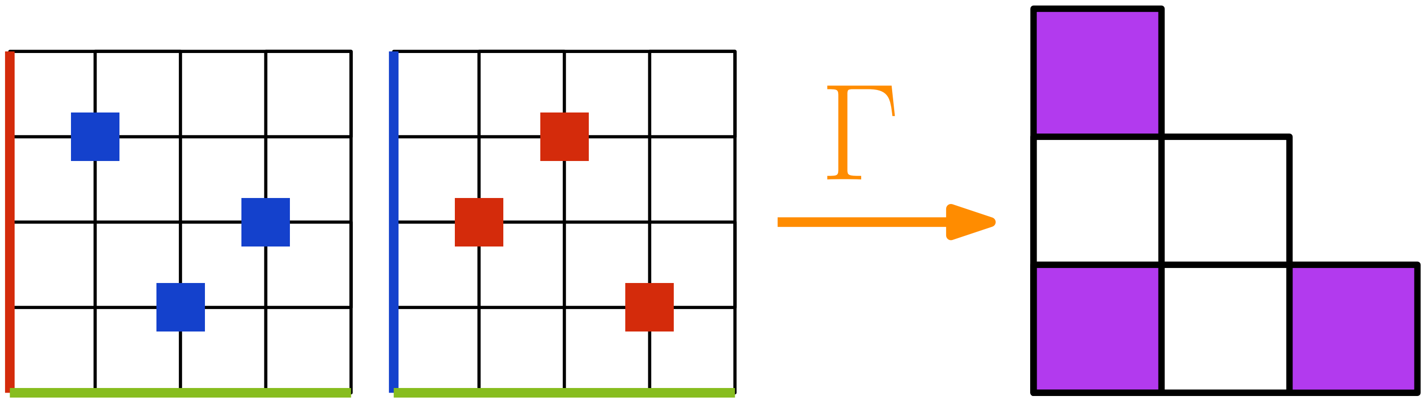


the only sparse configuration of size 3 that does not fill.

Intuition: Avoiding $(312, 231)$ prevents "gaps".

Avoiding $(312, 231)$: no "points too far"



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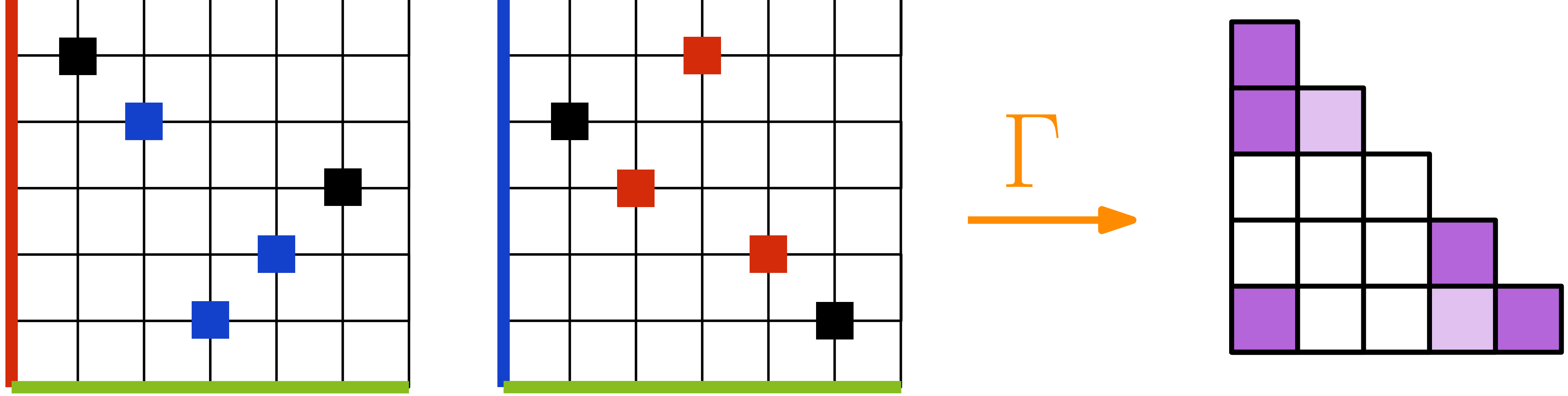

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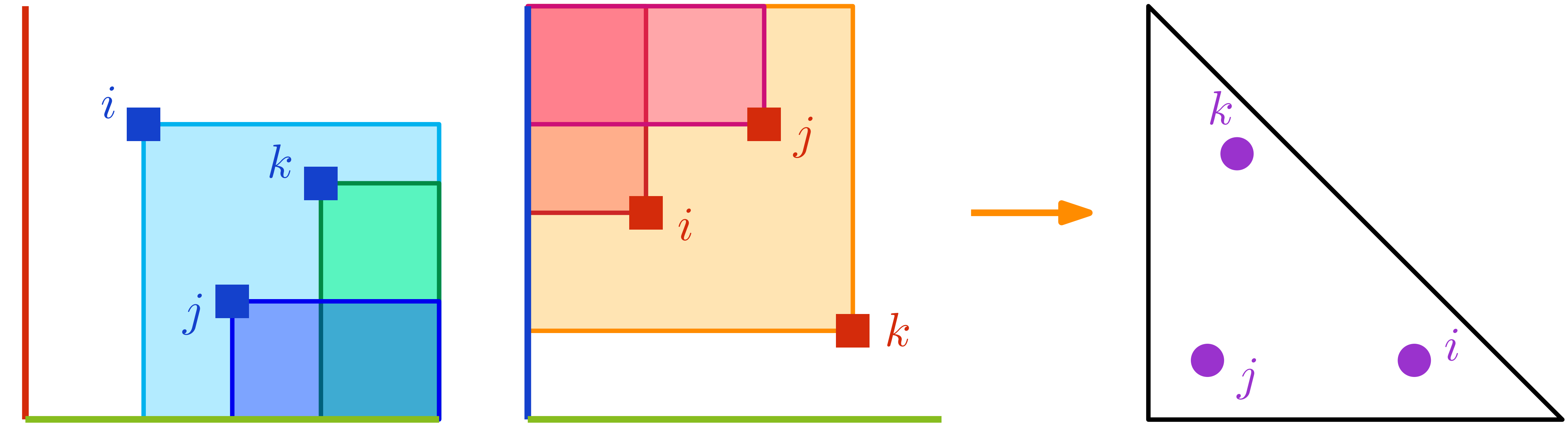
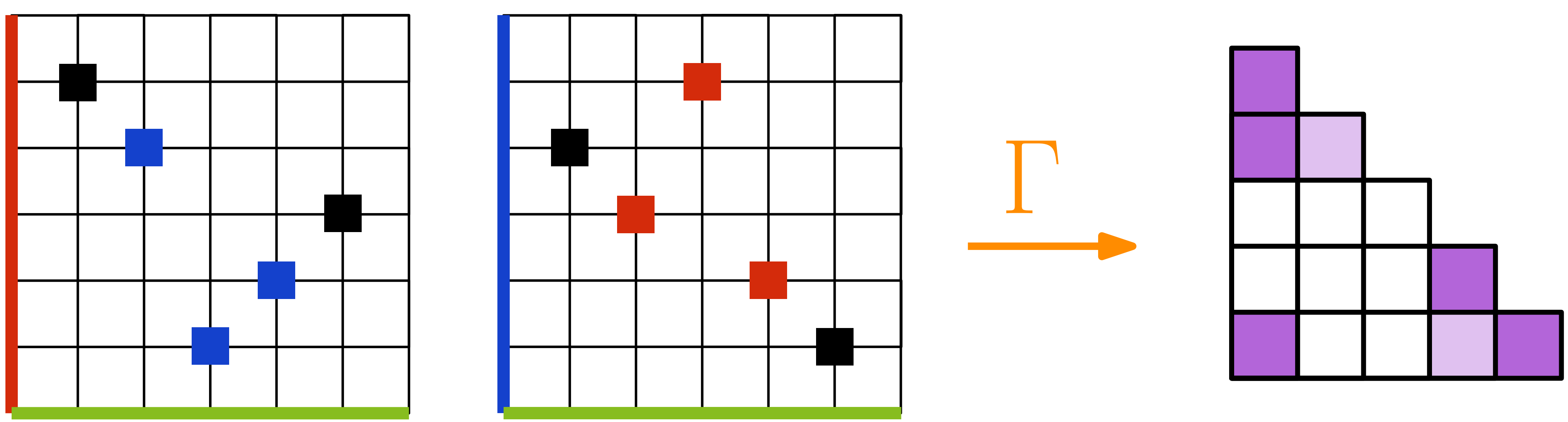


$r_\sigma(i) \gg r_\sigma(j) \approx r_\sigma(k)$

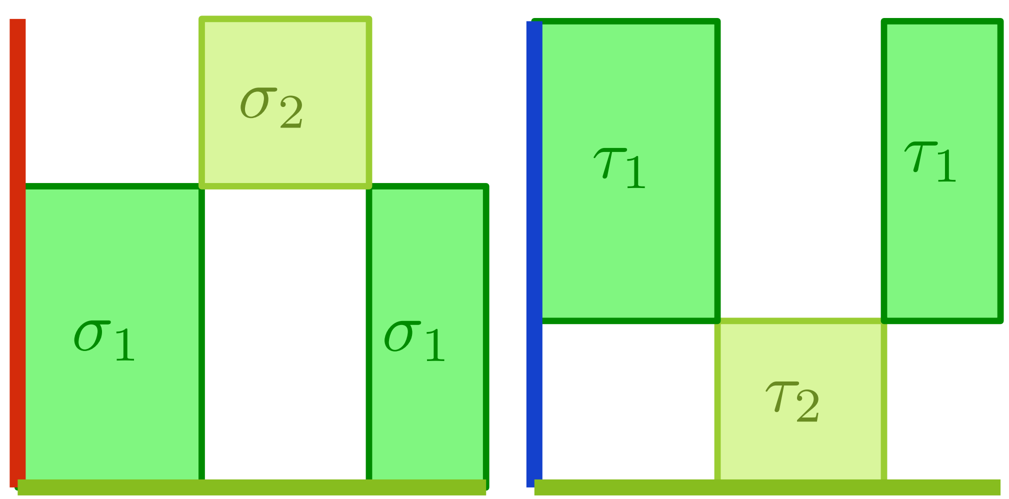
$\ell_\tau(i) \approx \ell_\tau(j) \ll \ell_\tau(k)$

Points too far to fill

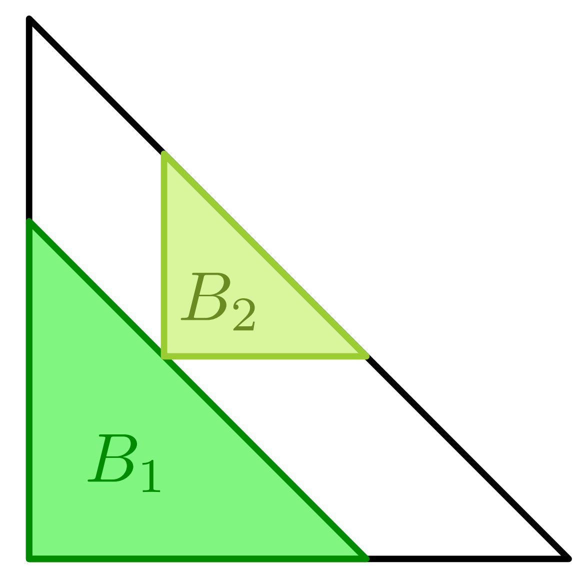


The key tool of the proof:

Isomorphic recursive decompositions



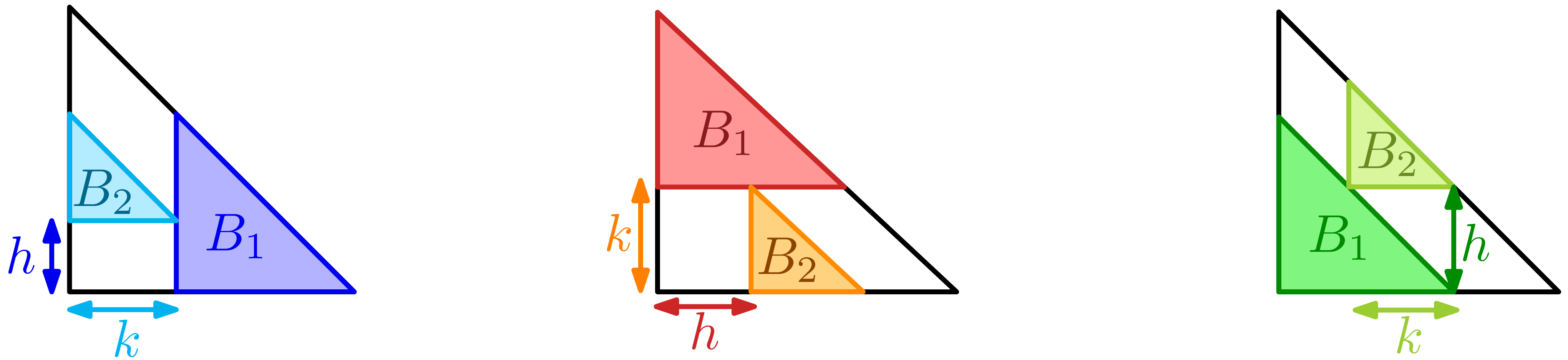
3-permutations



Bases

Isomorphic recursive decompositions

Lemma. [Salo, S. '22] Any basis of size $n \geqslant 2$ can be cut into two smaller bases in one of the 3 following ways.



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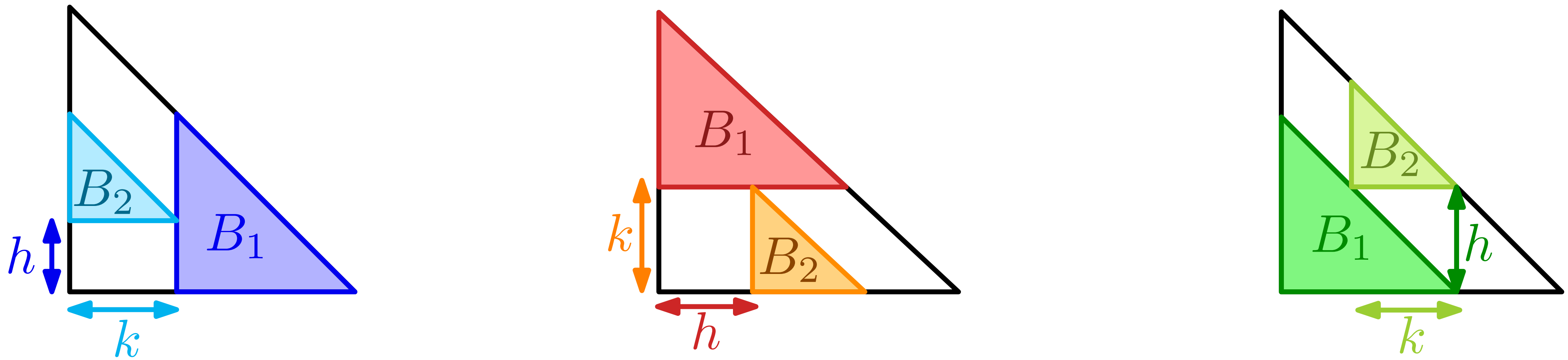
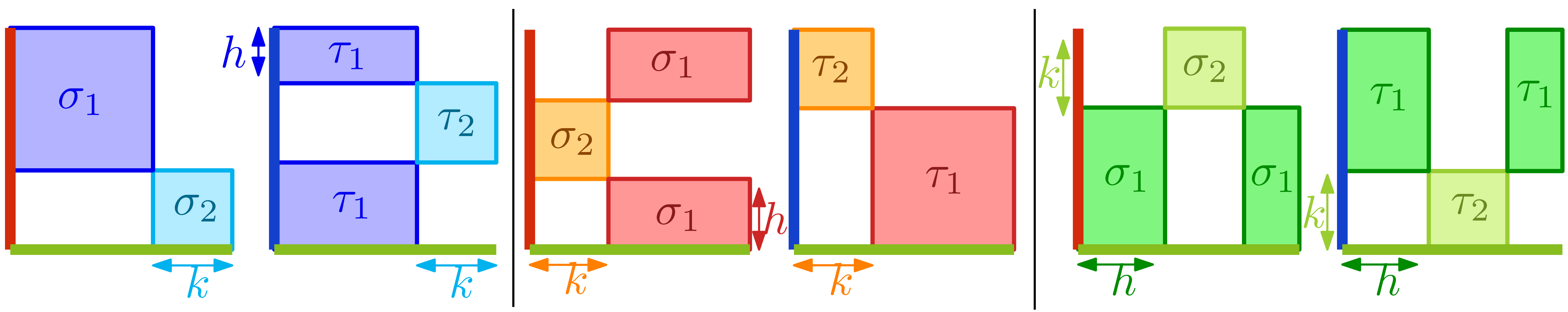


Lemma. [S. '25] Any 3-permutation of $Av_n((12, 12), (312, 231))$ can be cut into two smaller 3-permutations in one of the 3 following ways.



Proposition. [S. '25] Γ transports the cuts.

Isomorphic recursive decompositions

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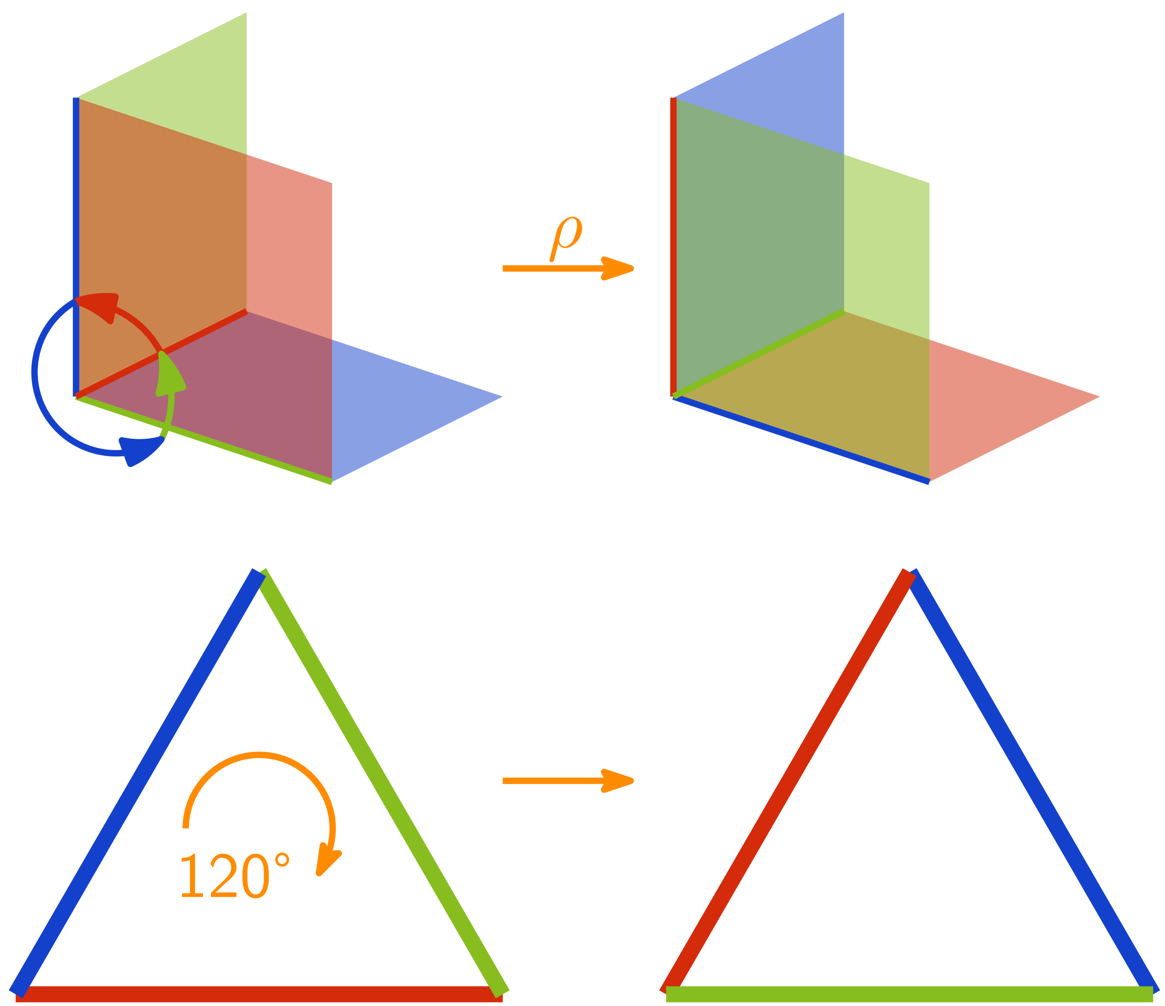
Proposition. [S. '25] Γ transports the cuts.

- We can now prove everything by induction!
  - $\Gamma(Av_n((12, 12), (312, 231))) \subset \mathcal{B}_n$
  - Γ is surjective
  - Γ is injective.

Nice properties and consequences

- Simple construction.
- Transports symmetries.



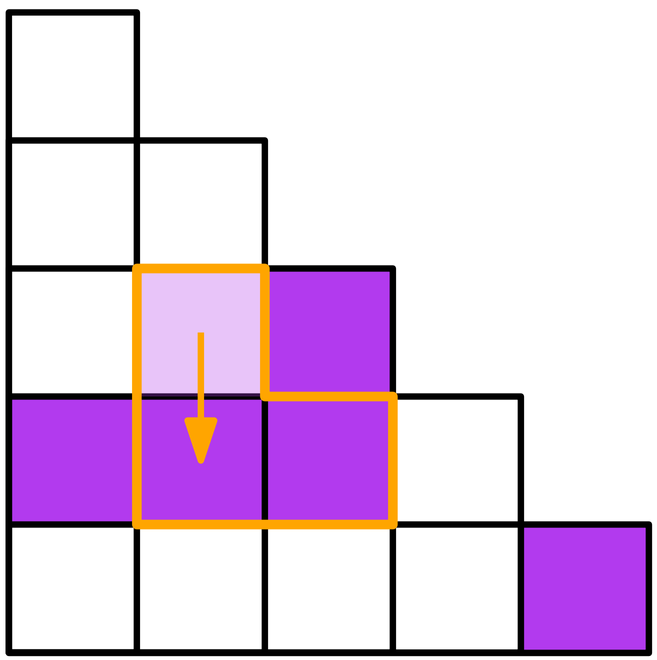
Nice properties and consequences

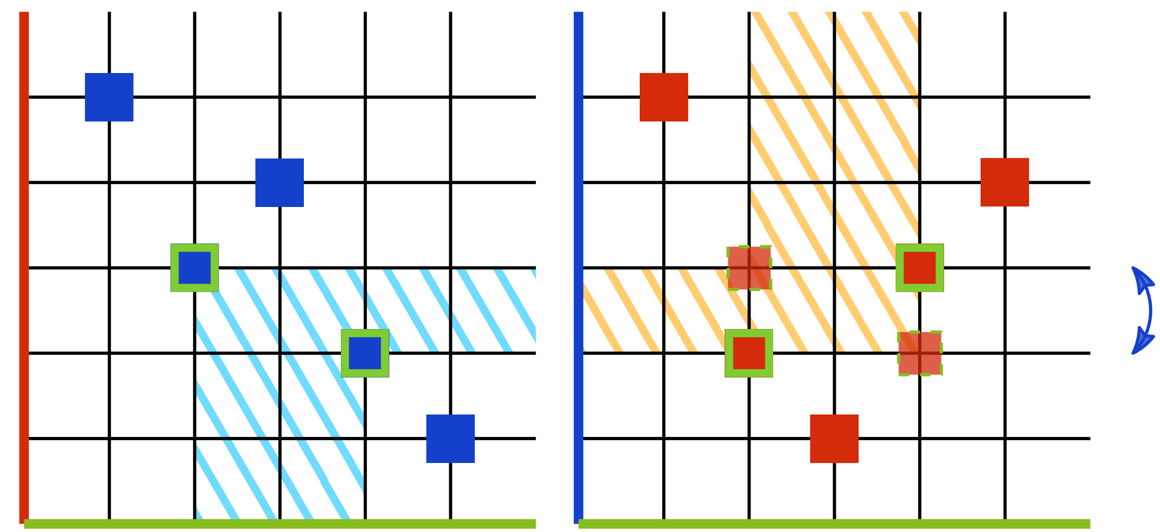
- Simple construction.
- Transports symmetries.
- Links two objects that are understood very differently $\Longrightarrow$ tools transfert.

  ▶ On bases:
  - A canonical labelling of the points.
  - Maybe a characterisation by forbidden patterns ?

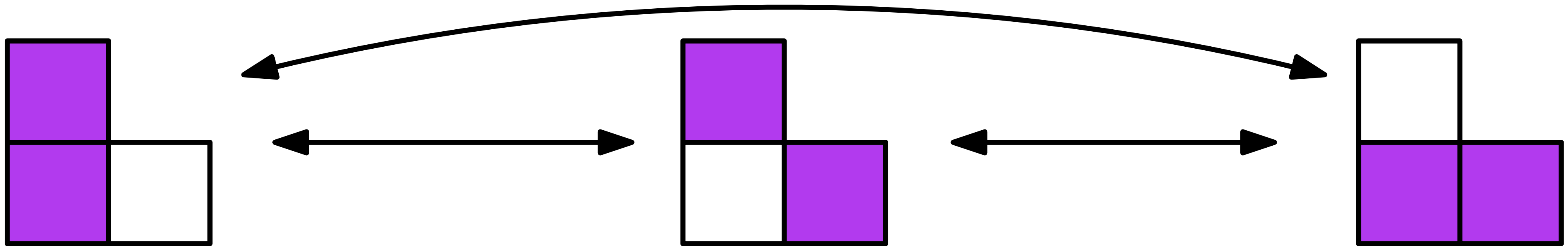
  ▶ On permutations: a dynamical system on 3-permutations (and others!) which could allow sampling.

III- Solitaire game





Solitaire game on configurations



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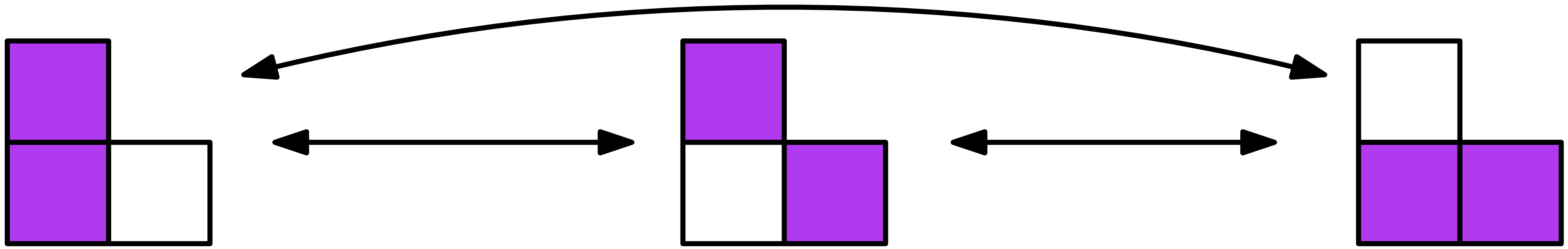

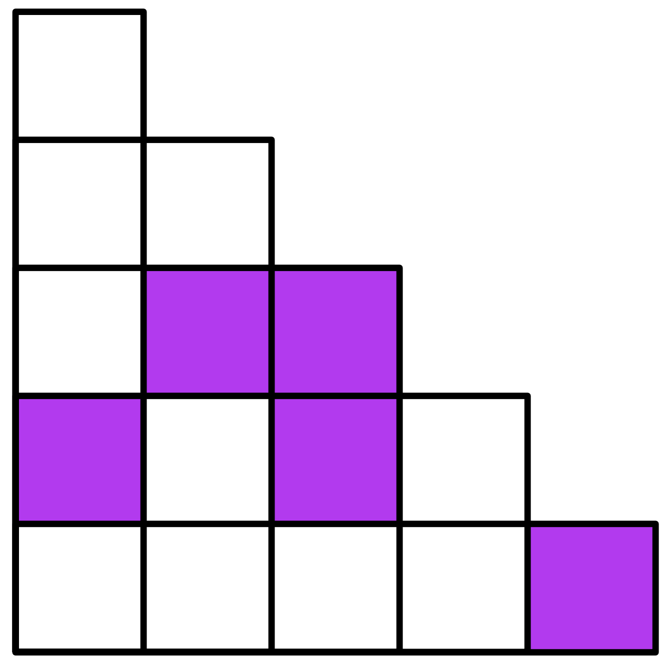

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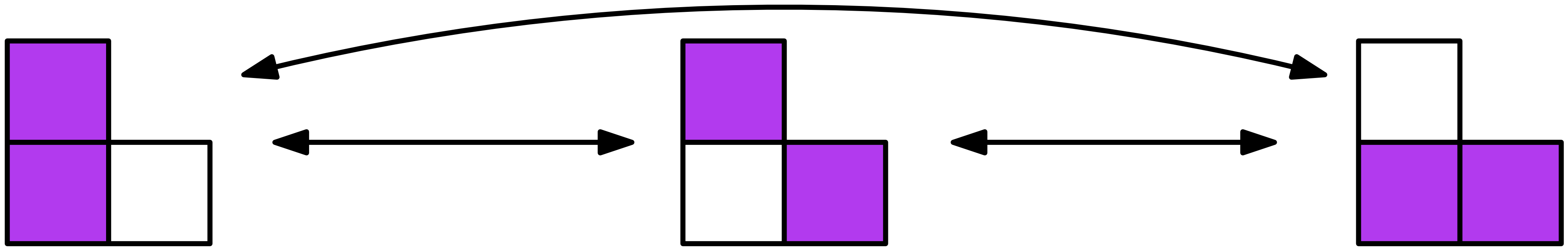

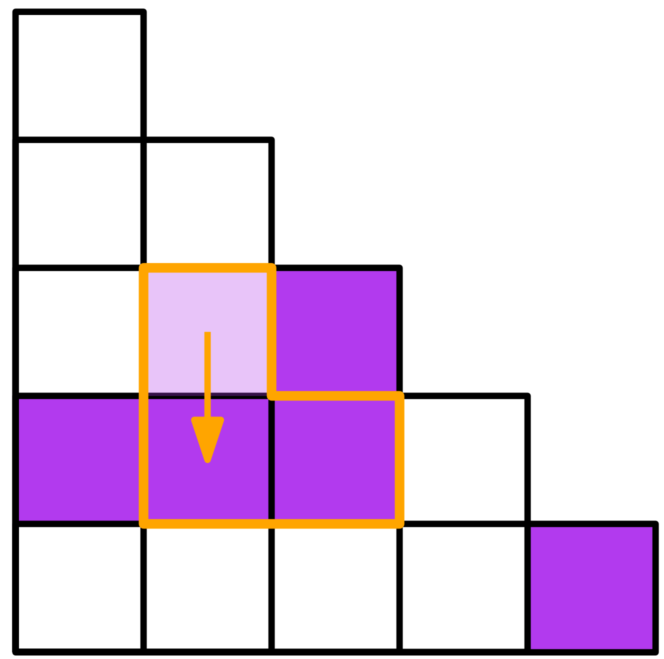

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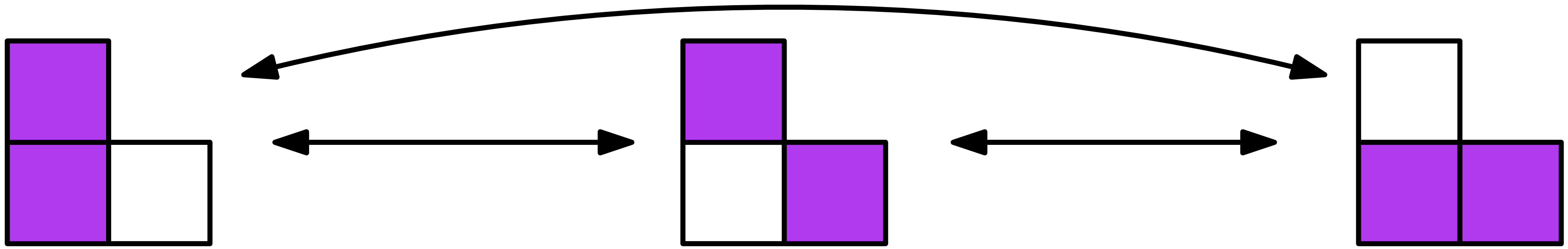

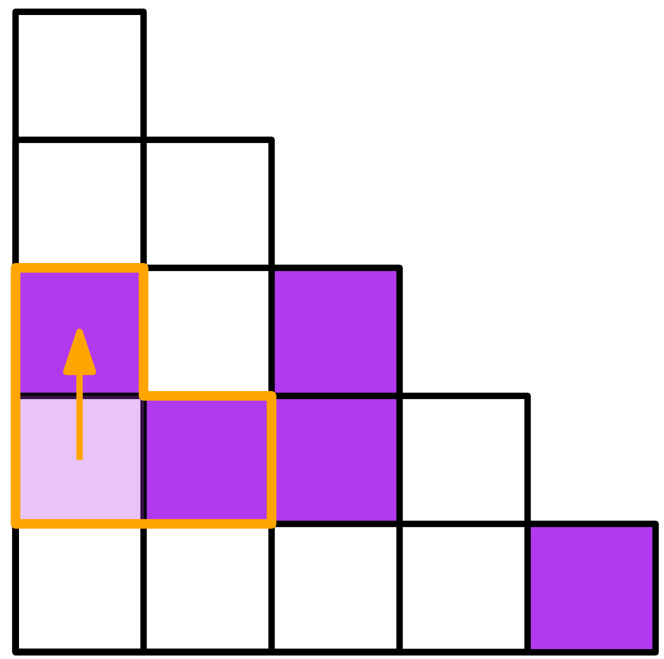

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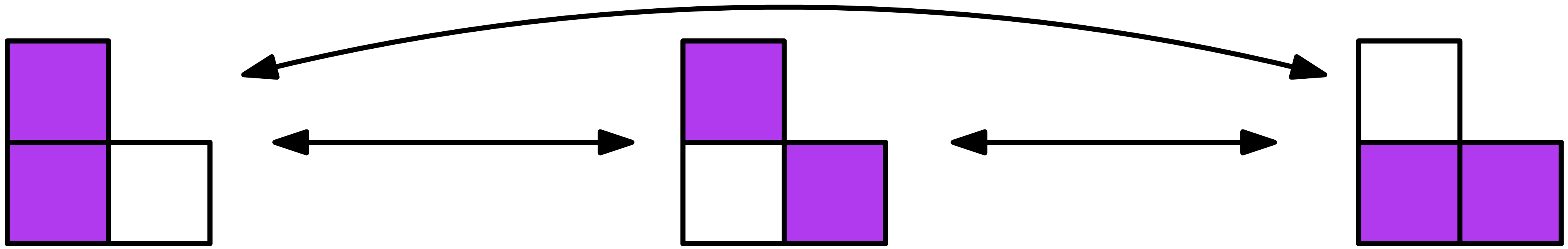

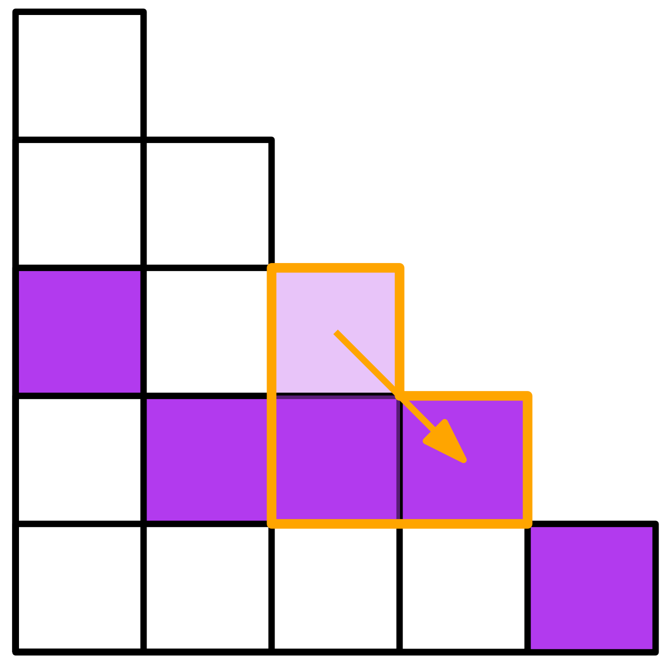

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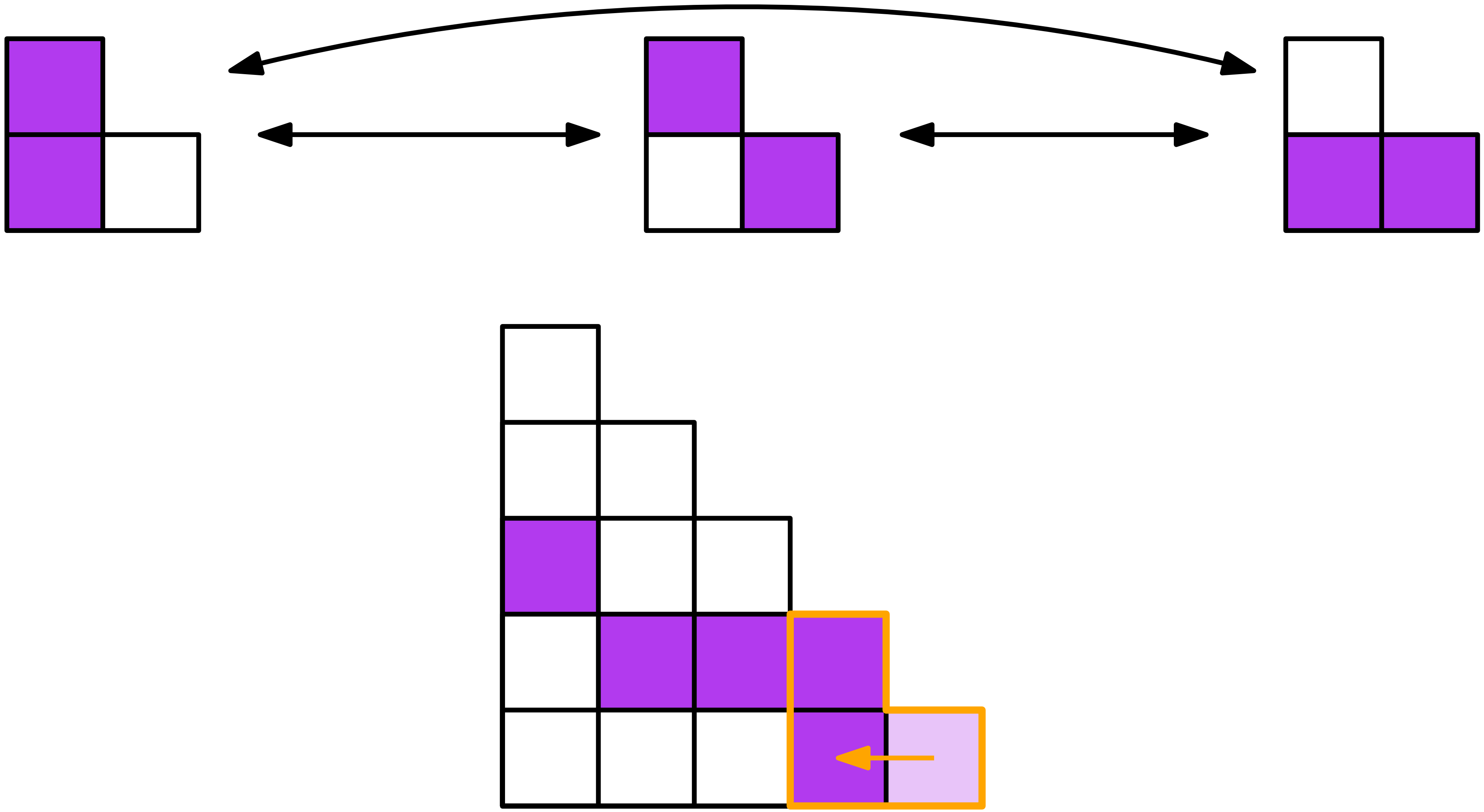

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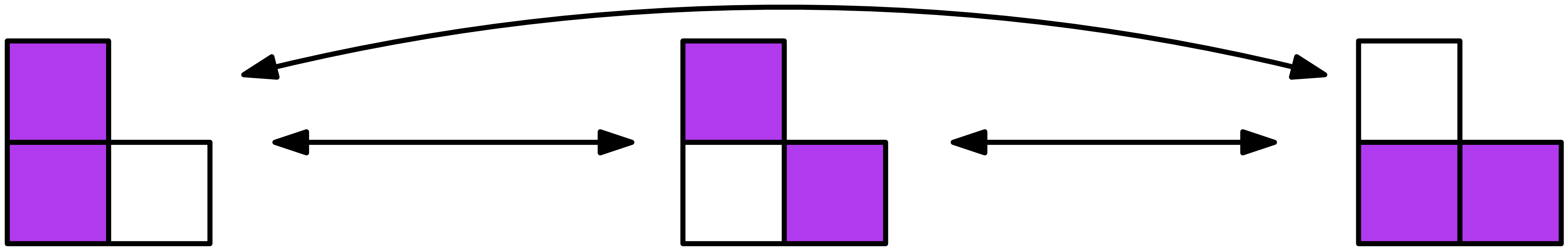

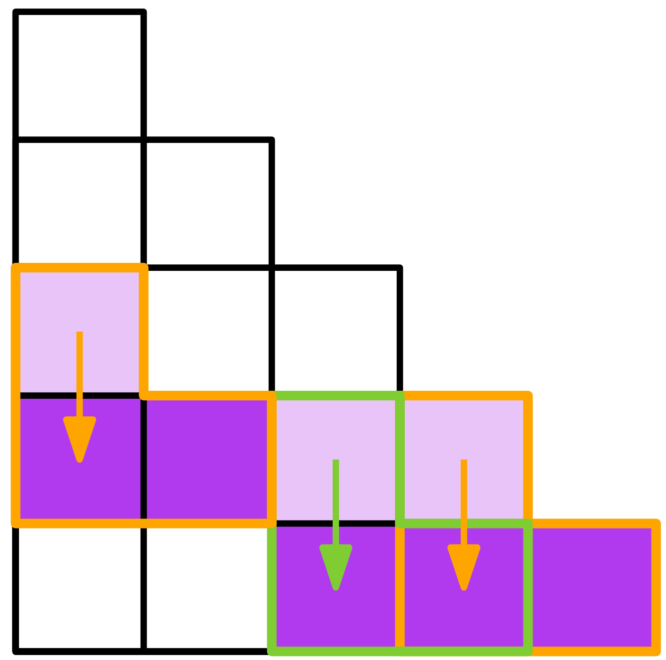

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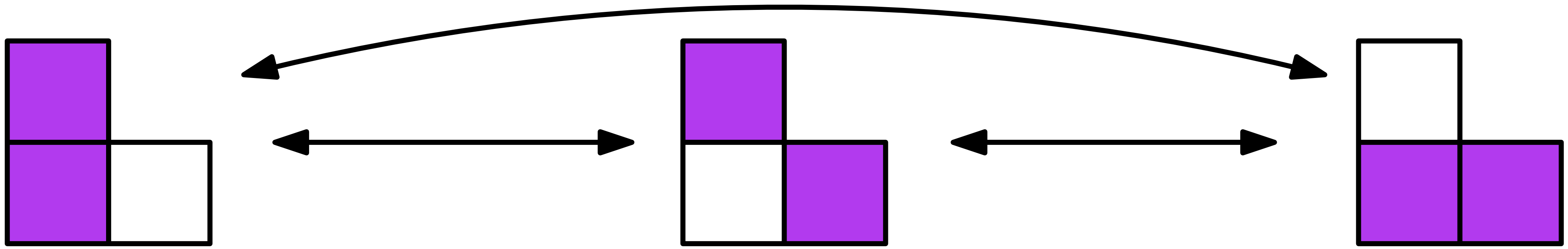

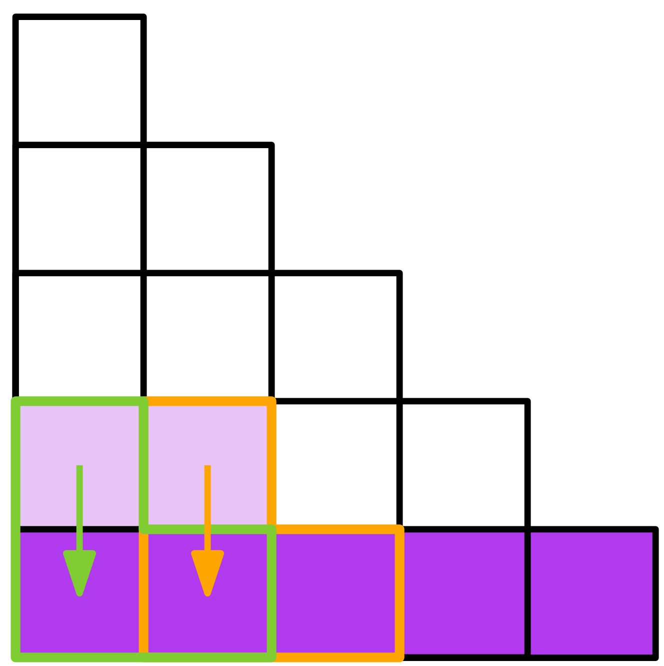

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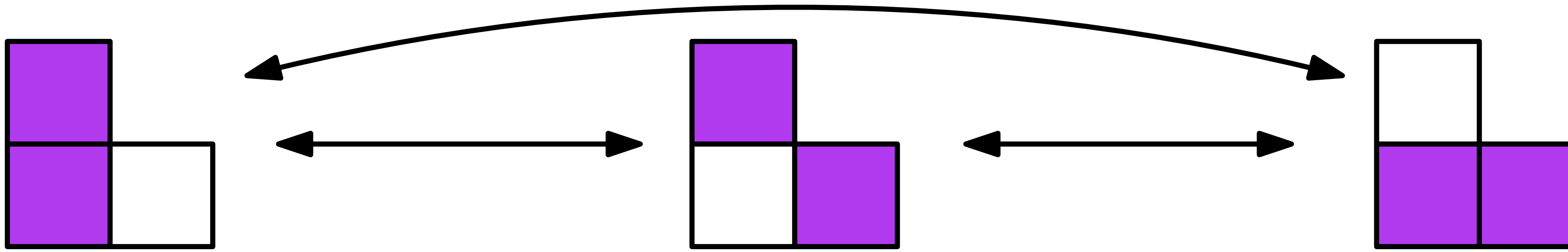
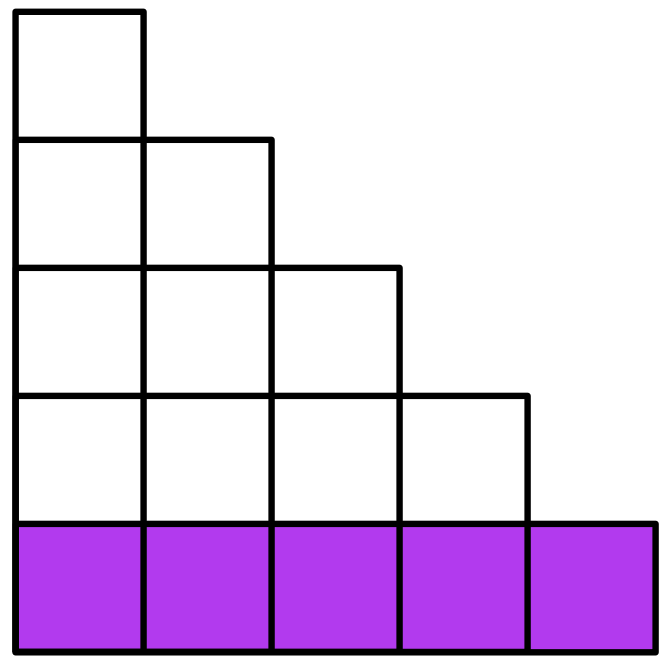
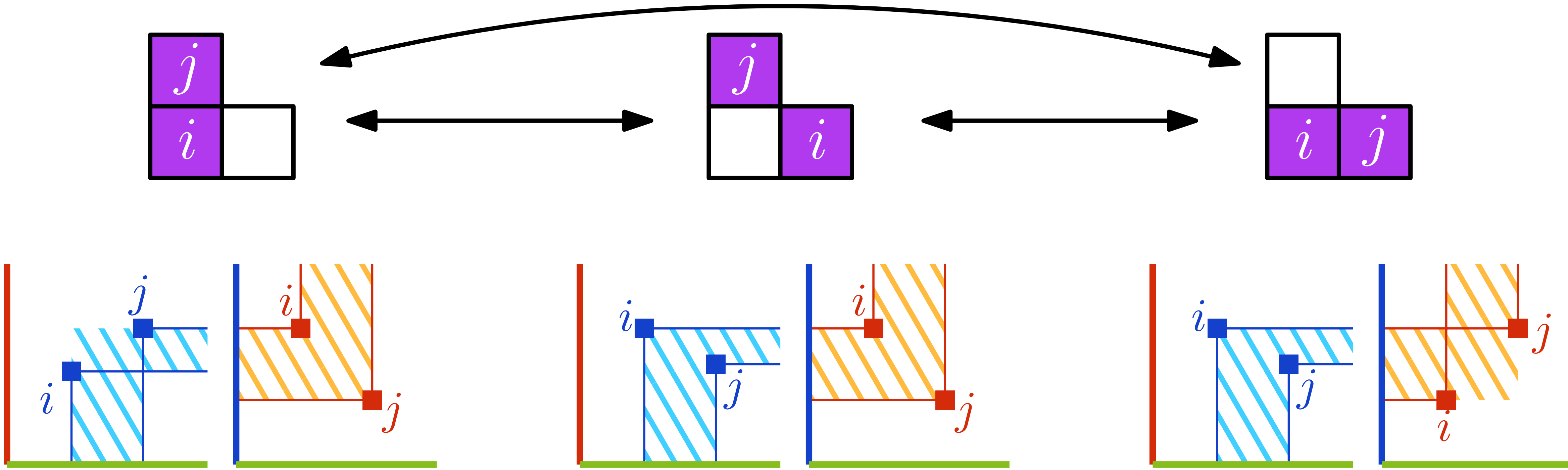




Theorem. [Salo, S. 22] The orbit of the line $[\![0, n-1]\!] \times \{0\}$ consists of the triangle bases of size n.

Solitaire game on permutations



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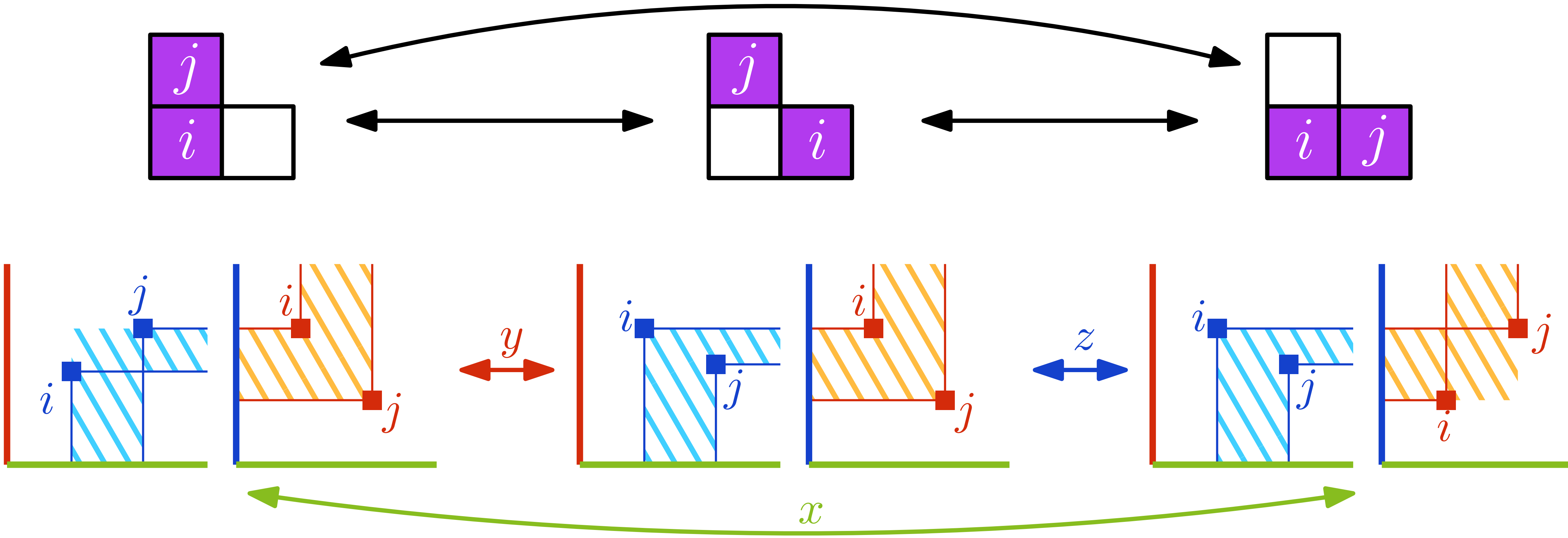

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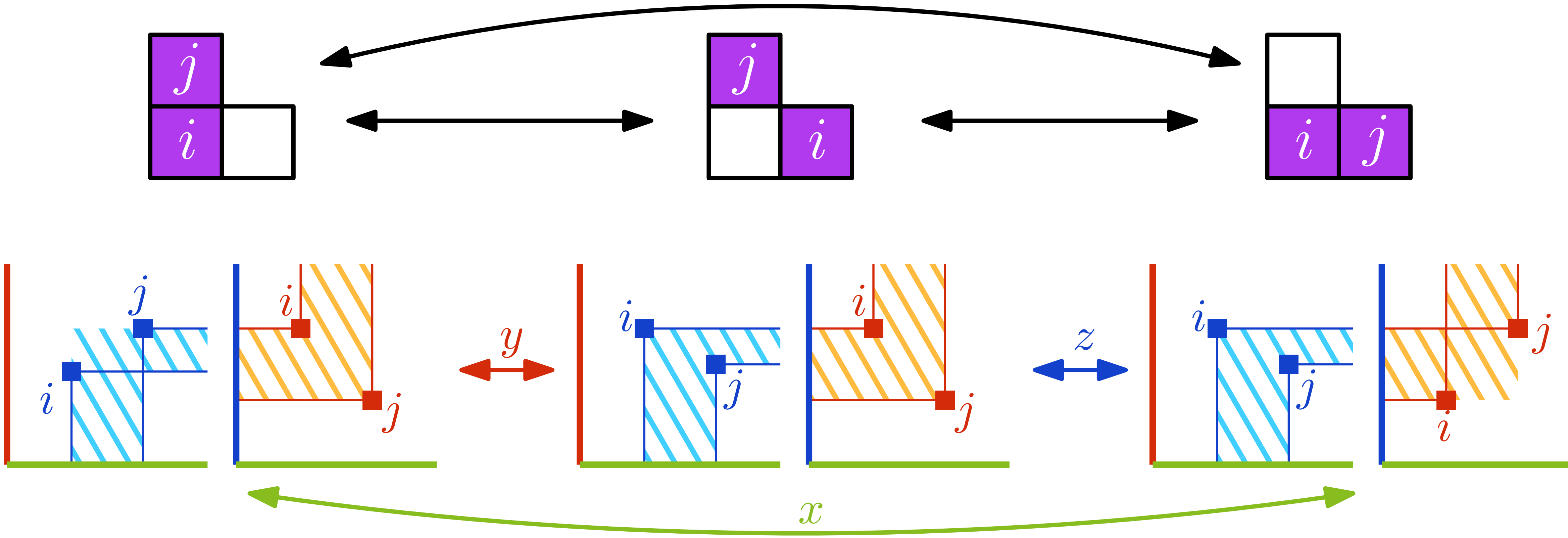
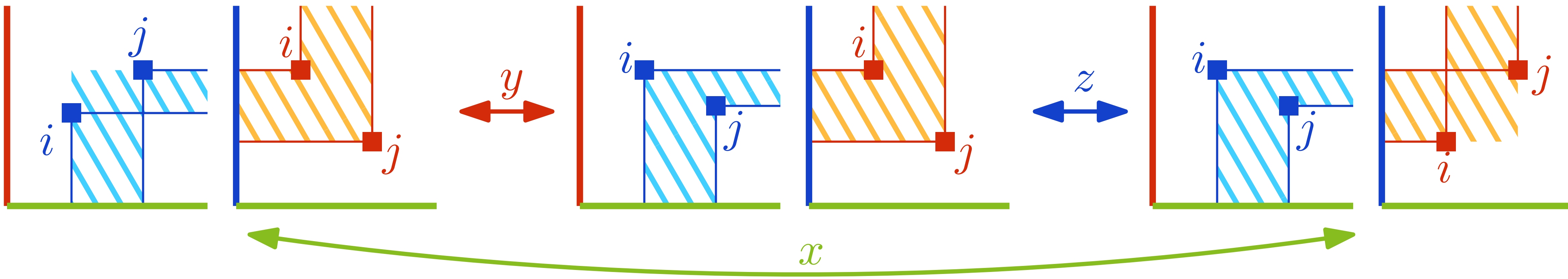
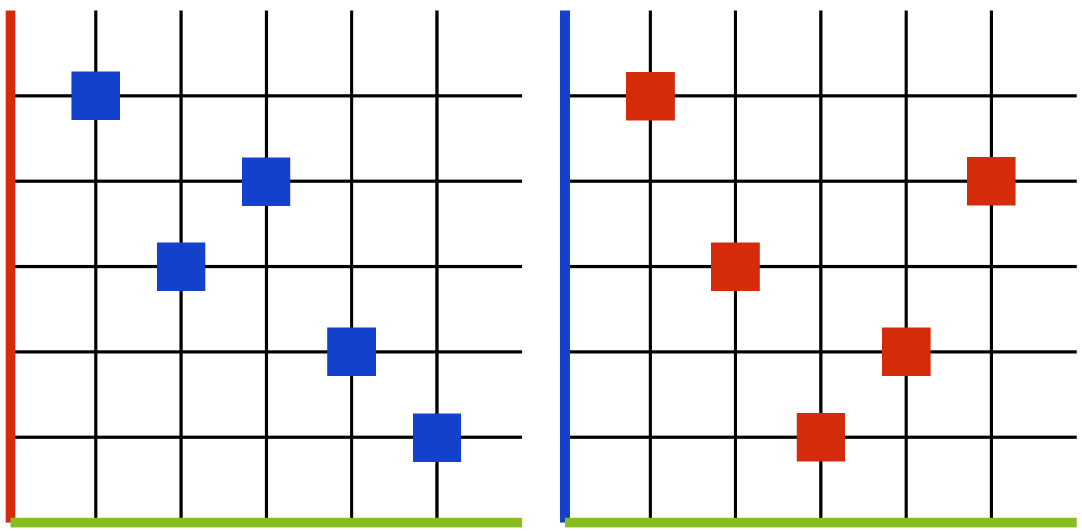


Theorem. [Salo, S. '22] The orbit of the line $[\![0, n-1]\!] \times \{0\}$ is the triangle bases of size n.

Theorem. The orbit of the line $(\overline{\text{id}}, \text{id})$ is $Av_n((12, 12), (312, 231))$.

Solitaire game on permutations





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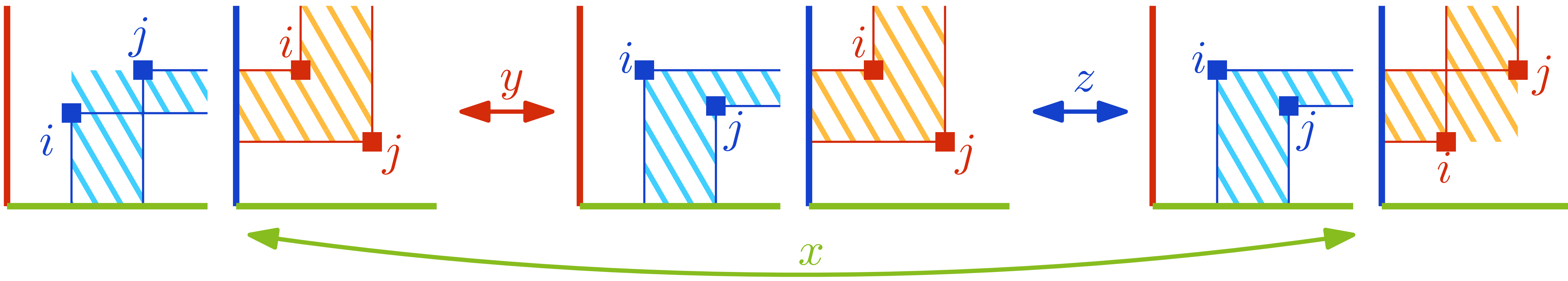

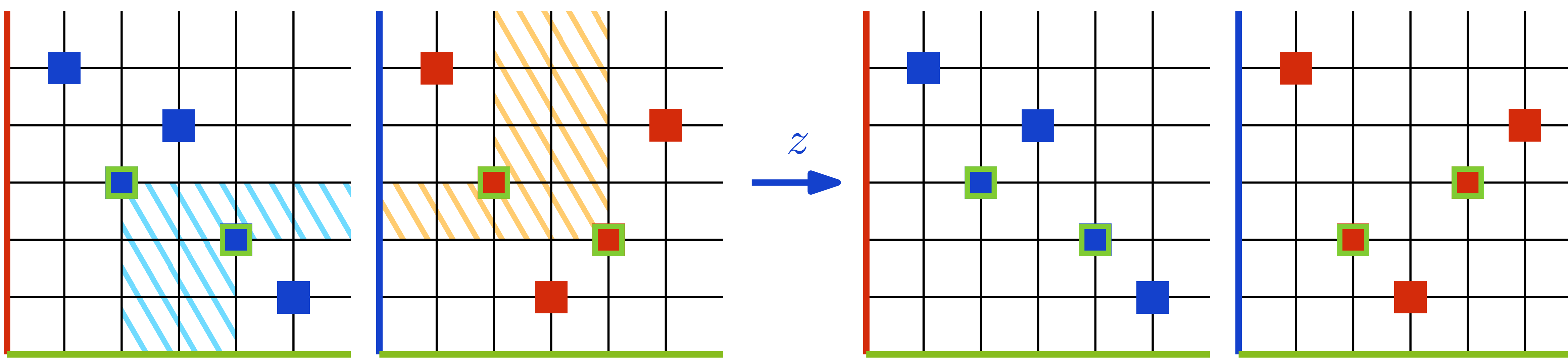

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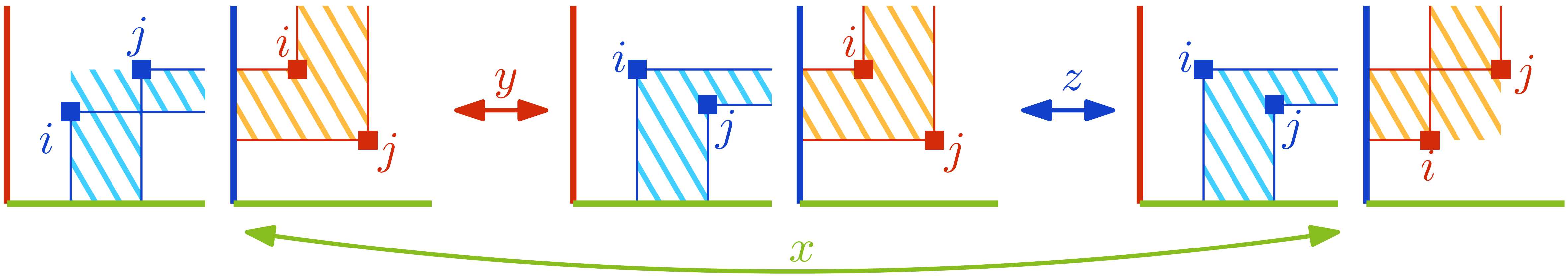

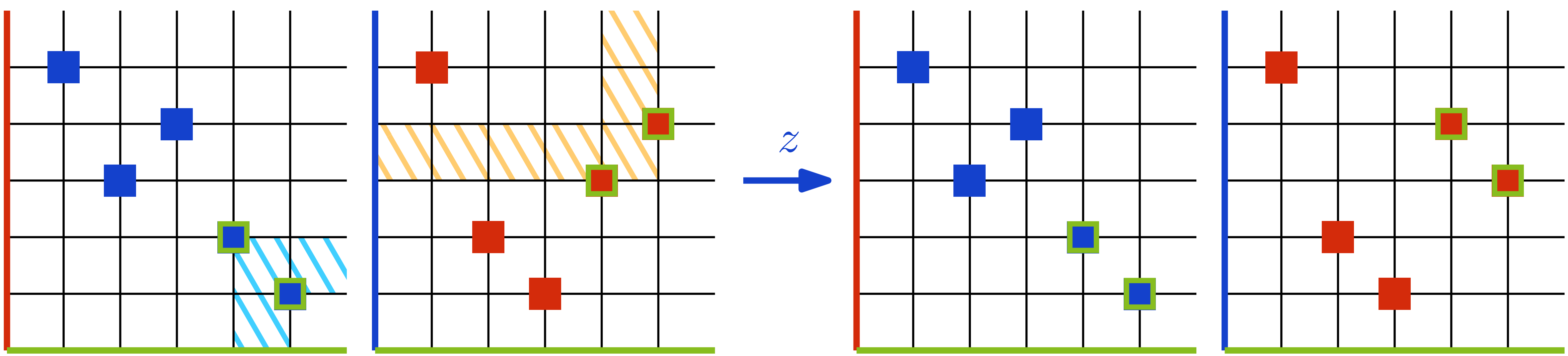

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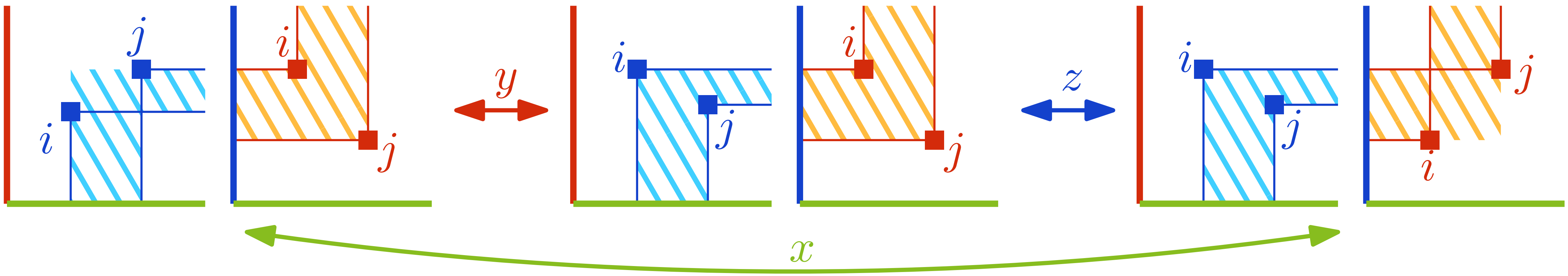

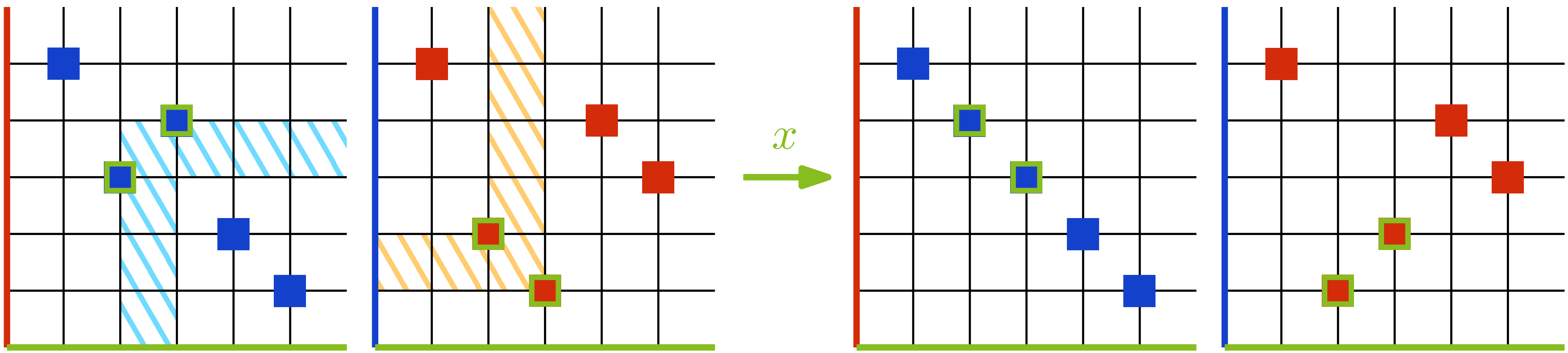

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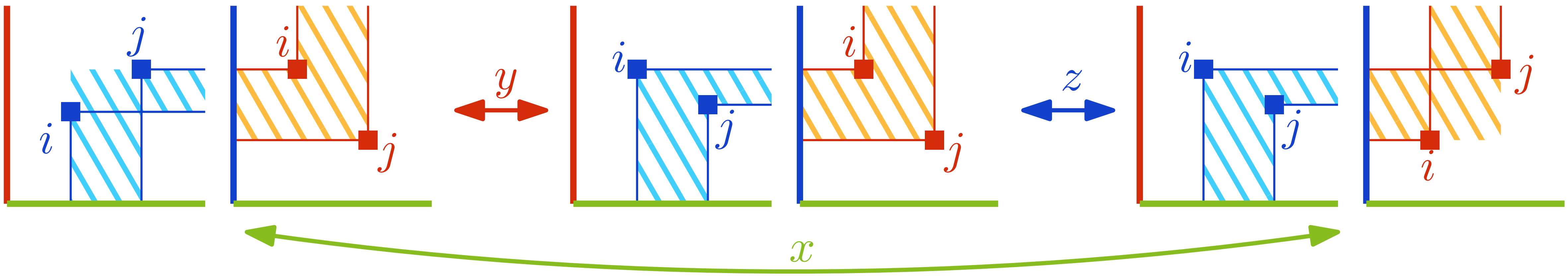

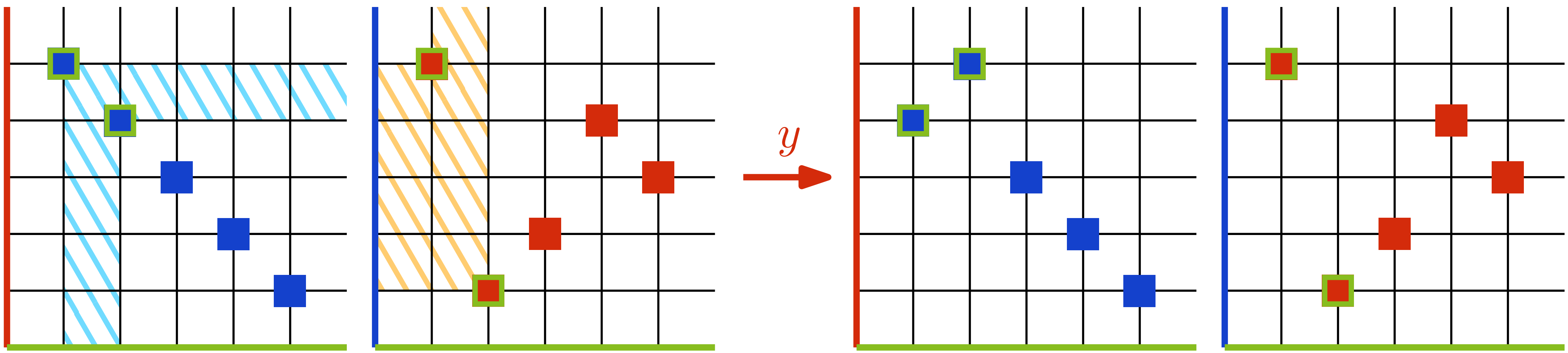

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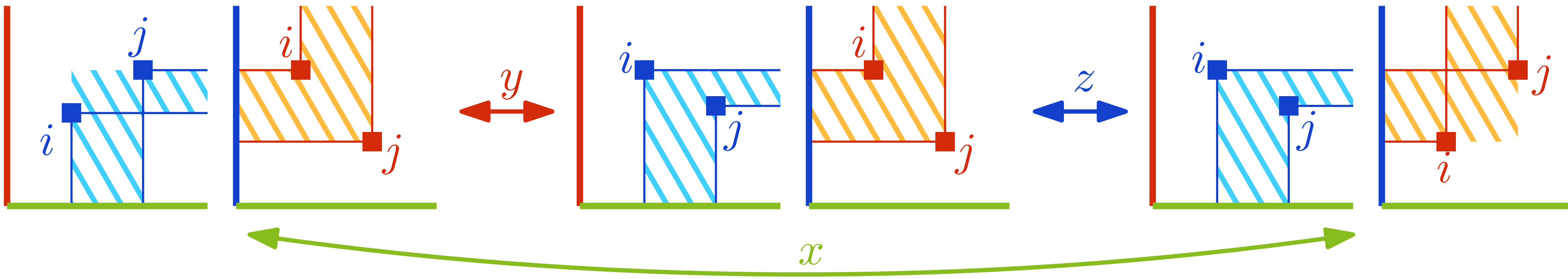

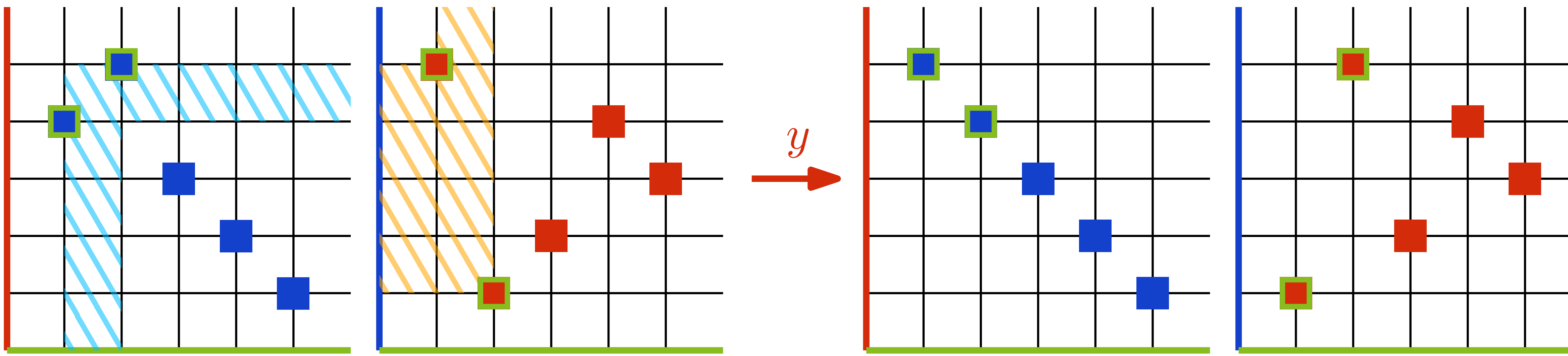

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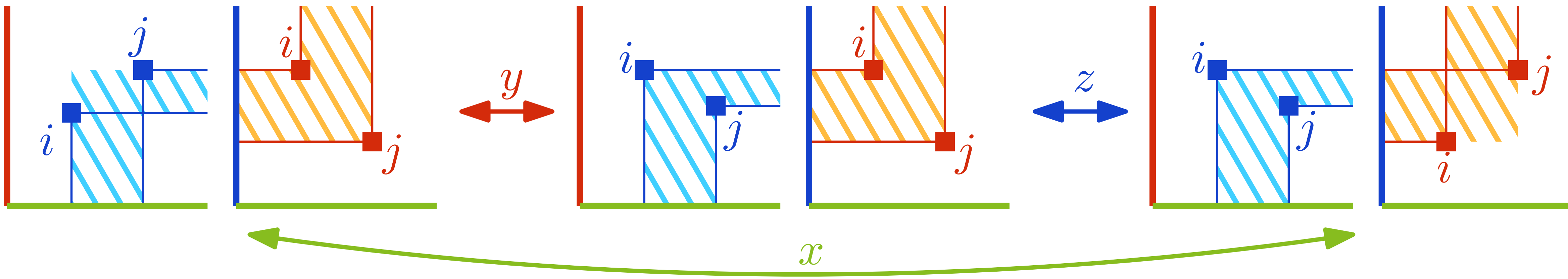

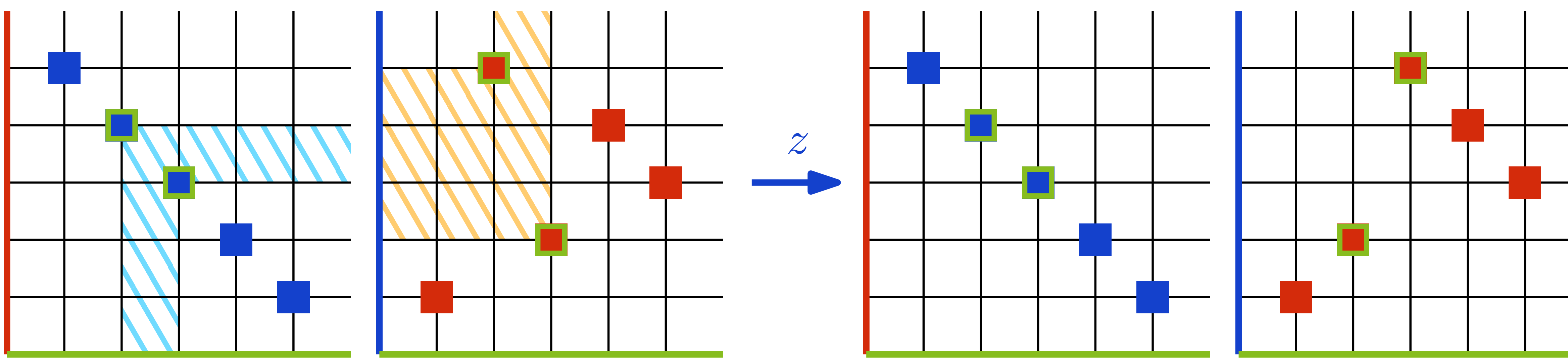

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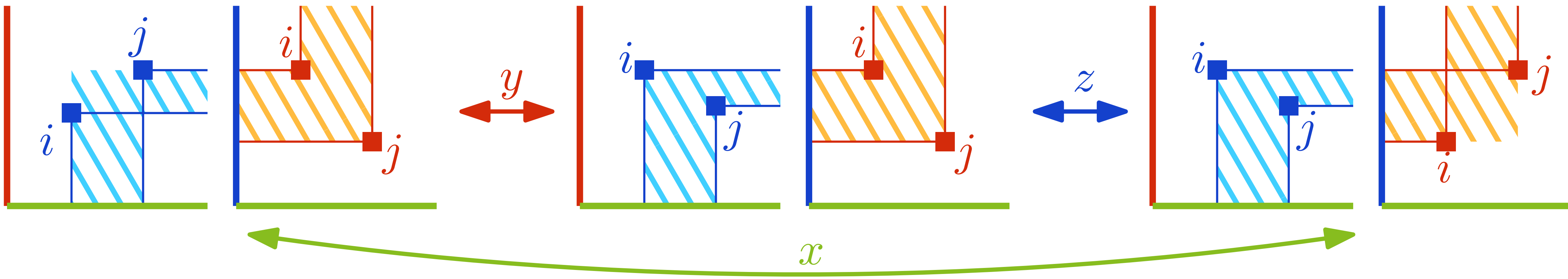

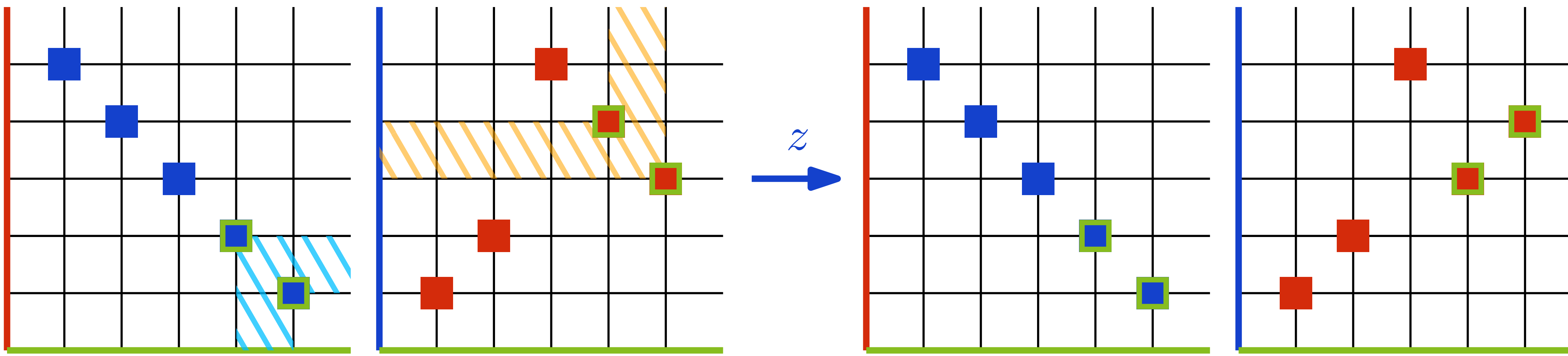

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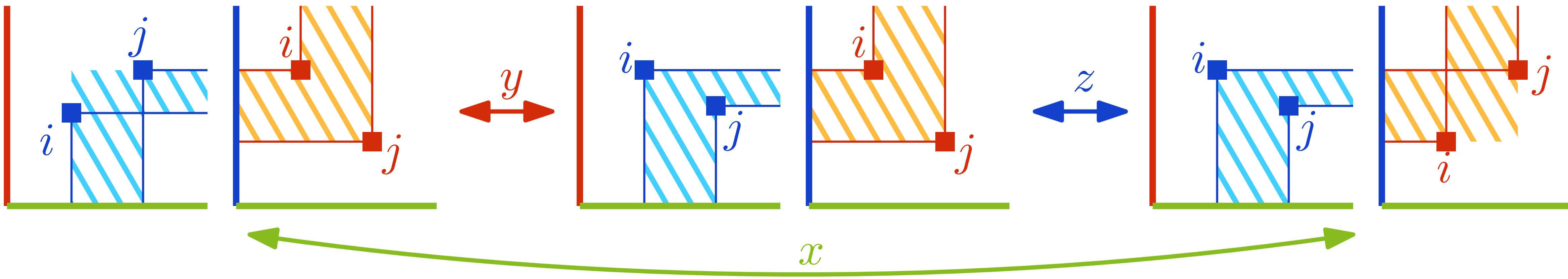

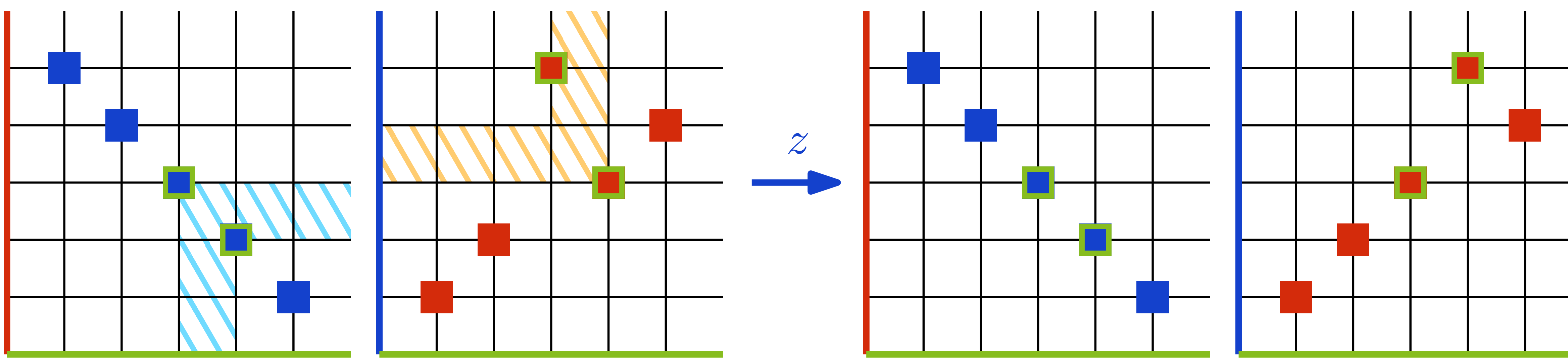

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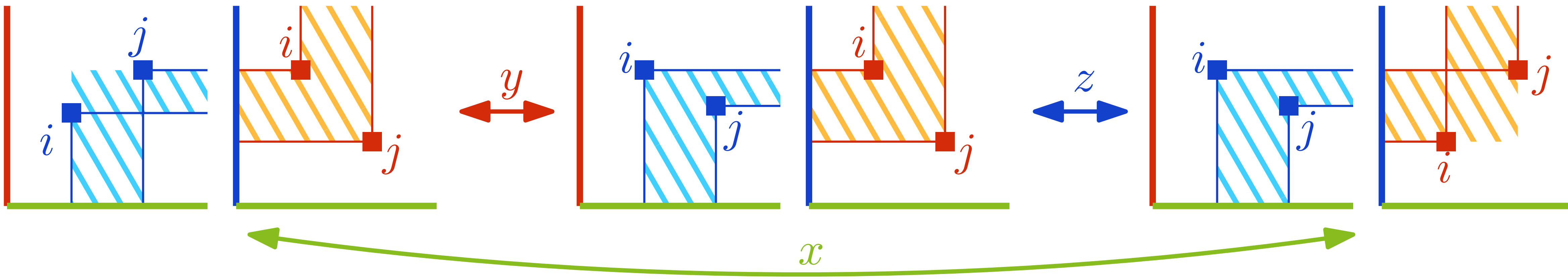

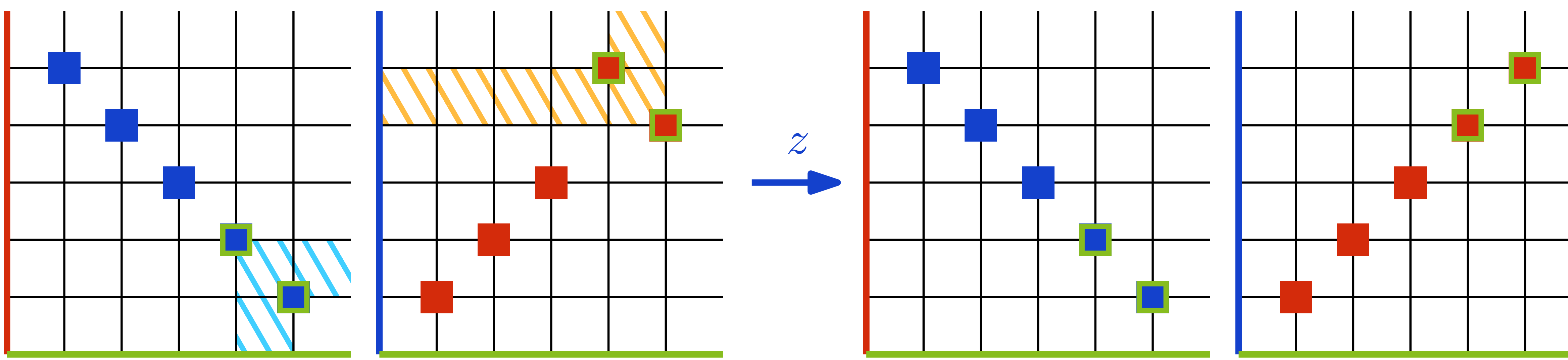

Uniform sampling?

Finding one basis is easy but enumerating them is hard $\Rightarrow$ use the solitaire for random sampling!

Question. What is the mixing time of the solitaire?

Lemma. [Salo, S. '22] The diameter of the line orbit for the solitaire is $\Theta(n^3)$.

What's next?

- No enumerative result.
  - Best known bounds : $3n! \leqslant |\mathcal{B}_n| \leqslant c\left(\frac{e}{2}\right)^n n^{n-\frac{5}{2}}$ with $c > 0$.

  $|Av_n((12, 12), (312, 231))| \leqslant |Av_n(12, 12)|$ = number of weak Bruhat intervals (unknown).
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- The solitaire is defined on all of $Av(12,12)$, what are the other orbits?
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Thank you!