On language stable subshifts

Samuel Petite
with V. Cyr, B. Kra

LAMFA UMR CNRS
Université de Picardie Jules Verne, France

June 19, 2025

Basic topological notions

Definition

Let (X, T) be a topological dynamical system, X a topological space. An automorphism $\phi\colon X \to X$ is an homeomorphism s.t.

$$\phi \circ T = T \circ \phi.$$

$$\mathrm{Aut}(X, T) = \{\phi \text{ automorphism of } (X, T)\}.$$

Basic topological notions

Definition

Let (X, T) be a topological dynamical system, X a topological space. An automorphism $\phi\colon X \to X$ is an homeomorphism s.t.

$$\phi \circ T = T \circ \phi.$$

$$\mathrm{Aut}(X, T) = \{\phi \text{ automorphism of } (X, T)\}.$$

$$\langle T \rangle \subset Z(\mathrm{Aut}(X, T)) \subset \mathrm{Aut}(X, T)$$

Basic topological notions

Definition

Let (X, T) be a topological dynamical system, X a topological space. An automorphism $\phi\colon X \to X$ is an homeomorphism s.t.

$$\phi \circ T = T \circ \phi.$$

$$\mathrm{Aut}(X, T) = \{\phi \text{ automorphism of } (X, T)\}.$$

$$\langle T \rangle \subset Z(\mathrm{Aut}(X, T)) \subset \mathrm{Aut}(X, T)$$

- Q: What can we say on $\mathrm{Aut}(X, T)$ as a group? Commutative? Amenable? What are the subgroups? the quotients?...
- Q: What do dynamical properties of (X, T) say about properties of $\mathrm{Aut}(X, T)$ and vice versa ?
- Q: How does $\mathrm{Aut}(X, T)$ act on X? On T-invariant measures?

Subshifts

Let A be a finite alphabet.
$A^{\mathbb{Z}}$ endowed with the product topology.
The shift map

$$\begin{array}{rcl} \sigma : A^{\mathbb{Z}} & \to & A^{\mathbb{Z}} \\ (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

For a closed set $X \subset A^{\mathbb{Z}}$, shift invariant ($\sigma(X) = X$), a subshift is the dynamical system $(X, \sigma_{|X})$.

Subshifts

Let A be a finite alphabet.
$A^{\mathbb{Z}}$ endowed with the product topology.
The shift map

$$\begin{array}{rcl} \sigma : A^{\mathbb{Z}} & \to & A^{\mathbb{Z}} \\ (x_n)_{n\in\mathbb{Z}} & \mapsto & (x_{n+1})_{n\in\mathbb{Z}} \end{array}$$

For a closed set $X \subset A^{\mathbb{Z}}$, shift invariant ($\sigma(X) = X$), a subshift is the dynamical system $(X, \sigma_{|X})$.
Similarly

$$X_{\mathcal{F}} = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{F}\ \forall m, i\}, \text{ where } \mathcal{F} \subset A^*.$$

Subshifts

$$X_{\mathcal{F}} = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{F} \ \forall m, i\}, \text{ where } \mathcal{F} \subset A^*.$$

Example

Subshifts

$$X_{\mathcal{F}} = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{F} \ \forall m, i\}, \text{ where } \mathcal{F} \subset A^*.$$

Example

- subshift $X_{\mathcal{F}}$ of finite type (SFT): $\mathcal{F}$ is finite.
  Ex $\mathcal{F} = \{11\}$, golden mean shift

$$\{(x_n)_n \in \{0,1\}^{\mathbb{Z}}; x_i x_{i+1} \neq 11 \quad \forall i\}.$$

Subshifts

$$X_{\mathcal{F}} = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{F}\ \forall m, i\}, \text{ where } \mathcal{F} \subset A^*.$$

Example

- subshift $X_{\mathcal{F}}$ of finite type (SFT): $\mathcal{F}$ is finite.
  Ex $\mathcal{F} = \{11\}$, golden mean shift

$$\{(x_n)_n \in \{0,1\}^{\mathbb{Z}}; x_i x_{i+1} \neq 11 \quad \forall i\}.$$

- Sofic subshift: $\mathcal{F}$ is a regular language
  Ex $\mathcal{F} = \{01^n0; n \text{ is even}\}$ even shift

$$\{(x_n)_n \in \{0,1\}^{\mathbb{Z}}; x_i \cdots x_{i+2n+1} \neq 01^{2n}0 \quad \forall i, n\}.$$

Subshifts

$$X_{\mathcal{F}} = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{F}\ \forall m, i\}, \text{ where } \mathcal{F} \subset A^*.$$

Example

- subshift $X_{\mathcal{F}}$ of finite type (SFT): $\mathcal{F}$ is finite.
  Ex $\mathcal{F} = \{11\}$, golden mean shift
  $$\{(x_n)_n \in \{0,1\}^{\mathbb{Z}}; x_i x_{i+1} \neq 11 \quad \forall i\}.$$
- Sofic subshift: $\mathcal{F}$ is a regular language
  Ex $\mathcal{F} = \{01^n0; n \text{ is even}\}$ even shift
  $$\{(x_n)_n \in \{0,1\}^{\mathbb{Z}}; x_i \cdots x_{i+2n+1} \neq 01^{2n}0 \quad \forall i, n\}.$$
- Given a language $\mathcal{L} \subset A^*$ that is extendable and factorial,
  $$X(\mathcal{L}) = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \in \mathcal{L} \quad \forall m, i\}.$$

Theorem (Curtis-Hedlund-Lyndon)

Any automorphism ϕ of (X, σ) is a cellular automaton:
There exists a local map $\hat{\phi}\colon A^{2r+1} \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \cdots x_{n+r}) \text{ for any } n \in \mathbb{Z}.$$

Theorem (Curtis-Hedlund-Lyndon)

Any automorphism ϕ of (X, σ) is a cellular automaton:
There exists a local map $\hat{\phi}\colon A^{2r+1} \to A$ *s.t.*

$$\phi(x)_n = \hat{\phi}(x_{n-r} \cdots x_{n+r}) \textit{ for any } n \in \mathbb{Z}.$$

Corollary

$Aut(X, \sigma)$ *is countable.*
$Aut(X, \sigma)$ *is a discrete subgroup of Homeo(X) for the uniform convergence topology.*

Pb: Does it exist a subshift with no $\mathrm{Aut}(X, \sigma)$-invariant measure?

¿$\exists$ a (probability) measure μ; $\quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in \mathrm{Aut}(X, \sigma)$?

Pb: Does it exist a subshift with no $\mathrm{Aut}(X, \sigma)$-invariant measure?

¿$\exists$ a (probability) measure μ; $\quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in \mathrm{Aut}(X, \sigma)$?

If it exists, a such measure is called a characteristic measure.

Pb: Does it exist a subshift with no $\mathrm{Aut}(X, \sigma)$-invariant measure?

¿$\exists$ a (probability) measure μ; $\quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in \mathrm{Aut}(X, \sigma)$?

If it exists, a such measure is called a characteristic measure.

- A uniquely ergodic system admits a characteristic measure.

Pb: Does it exist a subshift with no $\mathrm{Aut}(X,\sigma)$-invariant measure?

¿$\exists$ a (probability) measure μ; $\quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in \mathrm{Aut}(X,\sigma)$?

If it exists, a such measure is called a characteristic measure.

- A uniquely ergodic system admits a characteristic measure.
- A unique measure of maximal entropy is characteristic.

Pb: Does it exist a subshift with no $\mathrm{Aut}(X, \sigma)$-invariant measure?

¿$\exists$ a (probability) measure μ; $\quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in \mathrm{Aut}(X, \sigma)$?

If it exists, a such measure is called a characteristic measure.

- A uniquely ergodic system admits a characteristic measure.
- A unique measure of maximal entropy is characteristic.
- Any zero entropy subshift admits a characteristic measure. Frisch-Tamuz (22)

Pb: Does it exist a subshift with no $\mathrm{Aut}(X, \sigma)$-invariant measure?

¿$\exists$ a (probability) measure μ; $\quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in \mathrm{Aut}(X, \sigma)$?

If it exists, a such measure is called a characteristic measure.

- A uniquely ergodic system admits a characteristic measure.
- A unique measure of maximal entropy is characteristic.
- Any zero entropy subshift admits a characteristic measure. Frisch-Tamuz (22)

Pb: Does it exist a subshift with no $\mathrm{Aut}(X, \sigma)$-invariant measure?

¿$\exists$ a (probability) measure μ; $\quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in \mathrm{Aut}(X, \sigma)$?

If it exists, a such measure is called a characteristic measure.

- A uniquely ergodic system admits a characteristic measure.
- A unique measure of maximal entropy is characteristic.
- Any zero entropy subshift admits a characteristic measure. Frisch-Tamuz (22)

Pb: find a (generic) family of subshifts with characteristic measure including the mentioned cases.

Minimal forbidden word

Idea: use notion of minimal forbidden word

Béal-Mignosi-Restivo-Sciortino (00)

Minimal forbidden word

Idea: use notion of minimal forbidden word

Béal-Mignosi-Restivo-Sciortino (00)

The language of X

$$\mathcal{L}(X) = \{x_i \cdots x_j; x \in X, i < j\}.$$

Definition

For a subshift X with set of forbidden words $\mathcal{F} \subset A^*$, a word $w \in \mathcal{F}$ is minimal forbidden if any proper subword of w lies in $\mathcal{L}(X)$.

Definition

For a subshift X with set of forbidden words $\mathcal{F} \subset A^*$, a word $w \in \mathcal{F}$ is minimal forbidden if any proper subword of w lies in $\mathcal{L}(X)$.

Any forbidden word in $\mathcal{F}$ contains a minimal one.

Definition

For a subshift X with set of forbidden words $\mathcal{F} \subset A^*$, a word $w \in \mathcal{F}$ is minimal forbidden if any proper subword of w lies in $\mathcal{L}(X)$.

Any forbidden word in $\mathcal{F}$ contains a minimal one.

If $u_0 \cdots u_n$ is a minimal forbidden word of X,

$$u_0 u_1 \cdots u_{n-1} u_n \notin \mathcal{L}(X)$$

Definition

For a subshift X with set of forbidden words $\mathcal{F} \subset A^*$, a word $w \in \mathcal{F}$ is **minimal forbidden** *if any proper subword of w lies in $\mathcal{L}(X)$.*

Any forbidden word in $\mathcal{F}$ contains a minimal one.

If $u_0 \cdots u_n$ is a minimal forbidden word of X,

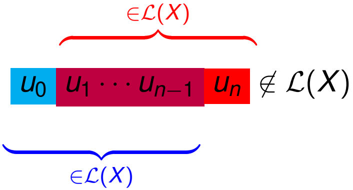
$$\underbrace{u_0 \; u_1 \cdots u_{n-1}}_{\in \mathcal{L}(X)} \; u_n \; \notin \mathcal{L}(X)$$

Definition

For a subshift X with set of forbidden words $\mathcal{F} \subset A^*$, a word $w \in \mathcal{F}$ is **minimal forbidden** *if any proper subword of w lies in $\mathcal{L}(X)$.*

Any forbidden word in $\mathcal{F}$ contains a minimal one.

If $u_0 \cdots u_n$ is a minimal forbidden word of X,



Definition

For a subshift X with set of forbidden words $\mathcal{F} \subset A^*$, a word $w \in \mathcal{F}$ is minimal forbidden if any proper subword of w lies in $\mathcal{L}(X)$.

Any forbidden word in $\mathcal{F}$ contains a minimal one.

If $u_0 \cdots u_n$ is a minimal forbidden word of X,

- The word $u_1 \cdots u_{n-1} \in \mathcal{L}(X)$ is the middle of the forbidden word $u_0 u_1 \cdots u_{n-1} u_n$.
- It is a bispecial word: i.e. $\exists a_1 \neq a_2, b_1 \neq b_2 \in A$ s.t.

$$a_1 u_1 \cdots u_{n-1} b_1 \text{ and } a_2 u_1 \cdots u_{n-1} b_2 \in \mathcal{L}(X)$$

Characterization of minimal forbidden words

The extension graph of $u \in \mathcal{L}(X)$ is the bipartite graph $\mathcal{E}(u)$ where

- left vertices are $\{a \in A; au \in \mathcal{L}(X)\}$;
- right vertices are $\{b \in A; ub \in \mathcal{L}(X)\}$;
- edges are $\{(a, b) \mid aub \in \mathcal{L}(X)\}$.

Characterization of minimal forbidden words

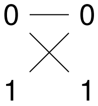
The extension graph of $u \in \mathcal{L}(X)$ is the bipartite graph $\mathcal{E}(u)$ where

- left vertices are $\{a \in A; au \in \mathcal{L}(X)\}$;
- right vertices are $\{b \in A; ub \in \mathcal{L}(X)\}$;
- edges are $\{(a, b) \mid aub \in \mathcal{L}(X)\}$.

Example

$\mathbf{x} = 01\underline{00101}\overline{00100}\underline{10100}101001001010010\cdots$

$\mathcal{E}(010)$



Characterization of minimal forbidden words

The extension graph of $u \in \mathcal{L}(X)$ is the bipartite graph $\mathcal{E}(u)$ where

- left vertices are $\{a \in A; au \in \mathcal{L}(X)\}$;
- right vertices are $\{b \in A; ub \in \mathcal{L}(X)\}$;
- edges are $\{(a, b) \mid aub \in \mathcal{L}(X)\}$.

Proposition

A word $u \in \mathcal{L}(X)$ is the middle of a minimal forbidden word $\iff$ its bipartite extension graph $\mathcal{E}(u)$ is not complete.

$\mathcal{M}(X)$ denote the set of minimal forbidden words of X.

$\mathcal{M}(X)$ denote the set of minimal forbidden words of X.

- For SFT, $\mathcal{M}(X_{\mathcal{F}}) \subset \mathcal{F}$ is finite.

  $\mathcal{M}(X) = \{11\}$ for the golden mean SFT

$\mathcal{M}(X)$ denote the set of minimal forbidden words of X.

- For SFT, $\mathcal{M}(X_{\mathcal{F}}) \subset \mathcal{F}$ is finite.

  $\mathcal{M}(X) = \{11\}$ for the golden mean SFT

- For sofic subshift, $\mathcal{M}(X)$ is a regular language

  Béal et al.

  $\mathcal{M}(X) = \{01^{2n}0; n \in \mathbb{N}\}$ for the even shift

$\mathcal{M}(X)$ denote the set of minimal forbidden words of X.

- For SFT, $\mathcal{M}(X_{\mathcal{F}}) \subset \mathcal{F}$ is finite.

  $\mathcal{M}(X) = \{11\}$ for the golden mean SFT

- For sofic subshift, $\mathcal{M}(X)$ is a regular language

  Béal et al.

  $\mathcal{M}(X) = \{01^{2n}0; n \in \mathbb{N}\}$ for the even shift

- For a general subshift X,

$$X = \{(x_n)_n \in A^{\mathbb{Z}}; x_i \cdots x_{i+m} \notin \mathcal{M}(X) \quad \forall m, i\}$$
$$\mathcal{L}(X) = A^* \setminus A^* \mathcal{M}(X) A^*$$

$\mathcal{M}(X)$ uniquely characterizes $\mathcal{L}(X)$.

Definition (Cyr-Kra)

A subshift X is language stable (LS) if the set

$$L\mathcal{M}(X) = \{n \in \mathbb{N}; \mathcal{M}(X) \cap A^n \neq \emptyset\}$$

has a zero lower uniform density, i.e.

$$\lim_{n \to +\infty} \min_{t \geq 0} \frac{1}{n} |L\mathcal{M}(X) \cap \{t+1, \dots, t+n\}| = 0.$$

Equivalently, the distance between 2 consecutive elements in $L\mathcal{M}(X)$ is unbounded.

Definition (Cyr-Kra)

A subshift X is language stable (LS) if the set

$$L\mathcal{M}(X) = \{n \in \mathbb{N}; \mathcal{M}(X) \cap A^n \neq \emptyset\}$$

has a zero lower uniform density, i.e.

$$\lim_{n \to +\infty} \min_{t \geq 0} \frac{1}{n} |L\mathcal{M}(X) \cap \{t+1, \ldots, t+n\}| = 0.$$

Equivalently, the distance between 2 consecutive elements in $L\mathcal{M}(X)$ is unbounded.

- SFT are language stable.

Definition (Cyr-Kra)

A subshift X is *language stable* *(LS) if the set*

$$L\mathcal{M}(X) = \{n \in \mathbb{N}; \mathcal{M}(X) \cap A^n \neq \emptyset\}$$

has a zero lower uniform density, i.e.

$$\lim_{n \to +\infty} \min_{t \geq 0} \frac{1}{n} |L\mathcal{M}(X) \cap \{t+1, \dots, t+n\}| = 0.$$

Equivalently, the distance between 2 consecutive elements in $L\mathcal{M}(X)$ is unbounded.

- SFT are language stable.
- Sofic shifts (not SFT) are **not** language stable.

Definition (Cyr-Kra)

A subshift X is language stable (LS) if the set

$$L\mathcal{M}(X) = \{n \in \mathbb{N}; \mathcal{M}(X) \cap A^n \neq \emptyset\}$$

has a zero lower uniform density, i.e.

$$\lim_{n\to+\infty} \min_{t\geq 0} \frac{1}{n}|L\mathcal{M}(X) \cap \{t+1, \ldots, t+n\}| = 0.$$

Equivalently, the distance between 2 consecutive elements in $L\mathcal{M}(X)$ is unbounded.

- Prouhet-Thue-Morse subshift is LS

Definition (Cyr-Kra)

A subshift X is language stable (LS) if the set

$$L\mathcal{M}(X) = \{n \in \mathbb{N}; \mathcal{M}(X) \cap A^n \neq \emptyset\}$$

has a zero lower uniform density, i.e.

$$\lim_{n \to +\infty} \min_{t \geq 0} \frac{1}{n} |L\mathcal{M}(X) \cap \{t+1, \dots, t+n\}| = 0.$$

Equivalently, the distance between 2 consecutive elements in $L\mathcal{M}(X)$ is unbounded.

- Prouhet-Thue-Morse subshift is LS $\tau : 1 \mapsto 10, 0 \mapsto 01$
  $\mathcal{L}(X) = \{\text{subword of } \tau^n(0), n \geq 0\}$.

Definition (Cyr-Kra)

A subshift X is language stable (LS) if the set

$$L\mathcal{M}(X) = \{n \in \mathbb{N}; \mathcal{M}(X) \cap A^n \neq \emptyset\}$$

has a zero lower uniform density, i.e.

$$\lim_{n\to+\infty} \min_{t\geq 0} \frac{1}{n} |L\mathcal{M}(X) \cap \{t+1, \ldots, t+n\}| = 0.$$

Equivalently, the distance between 2 consecutive elements in $L\mathcal{M}(X)$ is unbounded.

- Prouhet-Thue-Morse subshift is LS $\tau : 1 \mapsto 10, 0 \mapsto 01$
  $\mathcal{L}(X) = \{\text{subword of } \tau^n(0), n \geq 0\}$.
  Bispecial word are $\epsilon, 1, 0, \tau^n(10), \tau^n(01), \tau^n(101), \tau^n(010)$

Definition (Cyr-Kra)

A subshift X is language stable (LS) if the set

$$L\mathcal{M}(X) = \{n \in \mathbb{N}; \mathcal{M}(X) \cap A^n \neq \emptyset\}$$

has a zero lower uniform density, i.e.

$$\lim_{n \to +\infty} \min_{t \geq 0} \frac{1}{n} |L\mathcal{M}(X) \cap \{t+1, \ldots, t+n\}| = 0.$$

Equivalently, the distance between 2 consecutive elements in $L\mathcal{M}(X)$ is unbounded.

- Prouhet-Thue-Morse subshift is LS $\tau : 1 \mapsto 10, 0 \mapsto 01$
  $\mathcal{L}(X) = \{\text{subword of } \tau^n(0), n \geq 0\}$.
  Bispecial word are $\epsilon, 1, 0, \tau^n(10), \tau^n(01), \tau^n(101), \tau^n(010)$

$$L\mathcal{M}(X) \subset \{0, 1, 2^n, 2^n 3 \quad n \in \mathbb{N}\} + 2.$$

Idea under the definition

Idea under the definition

For any $n \in \mathbb{N}$, $X_n := X_{\cup_{\ell=1}^{n} \mathcal{M}(X) \cap A^\ell}$ is an SFT,

$$X = \bigcap_{n \geq 0} X_n.$$

The sequence $(X_n)_{n \geq 0}$ is the SFT cover of X.

Idea under the definition

For any $n \in \mathbb{N}$, $X_n := X_{\cup_{\ell=1}^{n} \mathcal{M}(X) \cap A^{\ell}}$ is an SFT,

$$X = \bigcap_{n \geq 0} X_n.$$

The sequence $(X_n)_{n \geq 0}$ is the SFT cover of X.

- $\mathcal{L}(X_{n+1}) \subset \mathcal{L}(X_n)$.

Idea under the definition

For any $n \in \mathbb{N}$, $X_n := X_{\cup_{\ell=1}^{n} \mathcal{M}(X) \cap A^\ell}$ is an SFT,

$$X = \bigcap_{n \geq 0} X_n.$$

The sequence $(X_n)_{n\geq 0}$ is the SFT cover of X.

- $\mathcal{L}(X_{n+1}) \subset \mathcal{L}(X_n)$.
- $\mathcal{L}(X_{n+1}) = \mathcal{L}(X_n)$ when $n+1 \notin L\mathcal{M}(X)$.

Idea under the definition

For any $n \in \mathbb{N}$, $X_n := X_{\cup_{\ell=1}^{n} \mathcal{M}(X) \cap A^\ell}$ is an SFT,

$$X = \bigcap_{n \geq 0} X_n.$$

The sequence $(X_n)_{n \geq 0}$ is the SFT cover of X.

- $\mathcal{L}(X_{n+1}) \subset \mathcal{L}(X_n)$.
- $\mathcal{L}(X_{n+1}) = \mathcal{L}(X_n)$ when $n+1 \notin L\mathcal{M}(X)$.

X is well approximated by SFT when X is language stable.

Theorem

The family of language stable subshifts is

- *invariant under conjugacies*

  Béal-Mignosi-Restivo-Sciortino (00)

- *generic*

  Cyr-Kra (21)

Complexity and language stable

Complexity $p_X(n) = |\mathcal{L}(X) \cap A^n|$, entropy $h = \lim_n \frac{\log(p_X(n))}{n}$.

Complexity and language stable

Complexity $p_X(n) = |\mathcal{L}(X) \cap A^n|$, entropy $h = \lim_n \frac{\log(p_X(n))}{n}$.

Proposition (CKP)

- Any entropy $h \geq 0$ is realizable by an LS subshift.

Complexity and language stable

Complexity $p_X(n) = |\mathcal{L}(X) \cap A^n|$, entropy $h = \lim_n \frac{\log(p_X(n))}{n}$.

Proposition (CKP)

- Any entropy $h \geq 0$ is realizable by an LS subshift.
- There exist LS subshifts with arbitrary polynomial complexity.

Complexity and language stable

Complexity $p_X(n) = |\mathcal{L}(X) \cap A^n|$, entropy $h = \lim_n \frac{\log(p_X(n))}{n}$.

Proposition (CKP)

- Any entropy $h \geq 0$ is realizable by an LS subshift.
- There exist LS subshifts with arbitrary polynomial complexity.
- If X is an aperiodic subshift with non-superlinear complexity, i.e.

$$\liminf_{n \to +\infty} \frac{p_X(n)}{n} < +\infty,$$

then it is language stable.

Complexity and language stable

Complexity $p_X(n) = |\mathcal{L}(X) \cap A^n|$, entropy $h = \lim_n \frac{\log(p_X(n))}{n}$.

Proposition (CKP)

- Any entropy $h \geq 0$ is realizable by an LS subshift.
- There exist LS subshifts with arbitrary polynomial complexity.
- If X is an aperiodic subshift with non-superlinear complexity, i.e.

$$\liminf_{n \to +\infty} \frac{p_X(n)}{n} < +\infty,$$

then it is language stable.

- $\exists$ non LS subshift with arbitrary entropy.

Complexity and language stable

Complexity $p_X(n) = |\mathcal{L}(X) \cap A^n|$, entropy $h = \lim_n \frac{\log(p_X(n))}{n}$.

Proposition (CKP)

- Any entropy $h \geq 0$ is realizable by an LS subshift.
- There exist LS subshifts with arbitrary polynomial complexity.
- If X is an aperiodic subshift with non-superlinear complexity, i.e.

$$\liminf_{n \to +\infty} \frac{p_X(n)}{n} < +\infty,$$

then it is language stable.

- $\exists$ non LS subshift with arbitrary entropy.
- $\exists$ non LS subshift with $n \log \log n$ complexity

$$\limsup_{n \to +\infty} \frac{p_X(n)}{n \log \log n} < +\infty.$$

Complexity and language stable

Proposition (CKP)

If X is a subshift with non-superlinear complexity, i.e.

$$\liminf_{n\to+\infty} \frac{p_X(n)}{n} < +\infty,$$

then it is language stable.

The proof based on :

Complexity and language stable

Proposition (CKP)

If X is a subshift with non-superlinear complexity, i.e.

$$\liminf_{n \to +\infty} \frac{p_X(n)}{n} < +\infty,$$

then it is language stable.

The proof based on :

- uniform bound on number of special words of a given length
- Fine and Wilf theorem (if X is aperiodic)

This provides the lengths of bispecial words form a zero density set.

Complexity and language stable

Complexity $p_X(n) = |\mathcal{L}(X) \cap A^n|$, entropy $h = \lim_n \frac{\log(p_X(n))}{n}$.

Proposition (CKP)

- Any entropy $h \geq 0$ is realizable by an LS subshift.
- There exist LS subshifts with arbitrary polynomial complexity.
- If X is an aperiodic subshift with non-superlinear complexity, i.e.

$$\liminf_{n \to +\infty} \frac{p_X(n)}{n} < +\infty,$$

  then it is language stable.

The set

$$\{\beta > 1;\ \text{the } \beta-\text{shift is LS}\} \text{ has full Lebesgue measure.}$$

Ergodic properties and Language Stable

A subshift X is minimal if it contains no proper subshift.

Ergodic properties and Language Stable

A subshift X is minimal if it contains no proper subshift.
Any sequence $x \in X$ is uniformly recurrent:

$$\forall n \in \mathbb{N}, \exists M_X(n) \in \mathbb{N} \text{ s.t.}$$
$$\forall v \in \mathcal{L}(X) \cap A^n \text{ occurs in any } u \in \mathcal{L}(X) \cap A^{M_X(n)}.$$

Ergodic properties and Language Stable

A subshift X is minimal if it contains no proper subshift.
Any sequence $x \in X$ is uniformly recurrent:

$$\forall n \in \mathbb{N}, \exists M_X(n) \in \mathbb{N} \text{ s.t.}$$
$$\forall v \in \mathcal{L}(X) \cap A^n \text{ occurs in any } u \in \mathcal{L}(X) \cap A^{M_X(n)}.$$

Proposition (CKP)

A minimal subshift X s.t.

$$M_X(n) = O(n^\alpha) \text{ for some } \alpha < \frac{1+\sqrt{3}}{2} = 1.36\ldots$$

is LS.

Ergodic properties and Language Stable

Proposition (CKP)

$\exists$ LS subshift X, where X is minimal and with arbitrary dimension group $C(X,\mathbb{Z})/\langle f - f \circ \sigma \rangle$.
In particular,

- the set of σ-invariant probability measure

$$\{\mu; \mu(\sigma^{-1}\cdot) = \mu(\cdot)\},$$

- the values $\mu(C)$ of clopens C,

can be chosen arbitrary.

Ergodic properties and Language Stable

Proposition (CKP)

$\exists$ LS subshift X, where X is minimal and with arbitrary dimension group $C(X, \mathbb{Z})/\langle f - f \circ \sigma \rangle$.
In particular,

- the set of σ-invariant probability measure

$$\{\mu; \mu(\sigma^{-1} \cdot) = \mu(\cdot)\},$$

- the values $\mu(C)$ of clopens C,

can be chosen arbitrary.

Proof is obtained thanks a (technical) condition on S-adic morphisms

- circular and biprefix
- growth condition on length of image of letters

Ergodic properties and Language Stable

Proposition (CKP)

Any speed-up of a minimal LS susbshift X, is LS:
For any continuous $p\colon X \to \mathbb{N}$, the system $(X, \sigma^{p(\cdot)}(\cdot))$ is LS.

e.g.: induced systems of minimal LS are LS.

Ergodic properties and Language Stable

Proposition (CKP)

Any speed-up of a minimal LS susbshift X, is LS:
For any continuous $p\colon X \to \mathbb{N}$, the system $(X, \sigma^{p(\cdot)}(\cdot))$ is LS.

e.g.: induced systems of minimal LS are LS.

Ergodic properties and Language Stable

Proposition (CKP)

Any speed-up of a minimal LS susbshift X, is LS:
For any continuous $p\colon X \to \mathbb{N}$, the system $(X, \sigma^{p(\cdot)}(\cdot))$ is LS.

e.g.: induced systems of minimal LS are LS.

$\exists$ minimal subshift that is not LS Pavlov

On automorphisms of LS subshifts

$$\mathrm{Aut}(X, \sigma) = \{\phi\colon X \to X; \phi \circ \sigma = \sigma \circ \phi\} \ni \sigma.$$

On automorphisms of LS subshifts

$$\mathrm{Aut}(X, \sigma) = \{\phi \colon X \to X; \phi \circ \sigma = \sigma \circ \phi\} \ni \sigma.$$

Theorem (Cyr-Kra)

If X is LS, then the $Aut(X, \sigma)$-action admits an invariant measure:

$$\exists \text{ measure } \mu; \quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in Aut(X, \sigma).$$

On automorphisms of LS subshifts

$$\mathrm{Aut}(X,\sigma) = \{\phi\colon X \to X; \phi \circ \sigma = \sigma \circ \phi\} \ni \sigma.$$

Theorem (Cyr-Kra)

If X is LS, then the $Aut(X,\sigma)$-action admits an invariant measure:

$$\exists \text{ measure } \mu; \quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in Aut(X,\sigma).$$

Theorem (Cyr-Kra-P)

Assume that X is LS and the gaps in $L\mathcal{M}(X)$ growth fast enough (explicit)
Then for any factor Y of X the $Aut(Y,\sigma)$-action admits an invariant measure:

$$\exists \text{ measure } \mu; \quad \mu(\phi^{-1}(\cdot)) = \mu(\cdot) \quad \forall \phi \in Aut(Y,\sigma).$$

Restrictions on LS subshifts

$$\mathrm{Aut}(X,\sigma) = \{\phi\colon X \to X; \phi \circ \sigma = \sigma \circ \phi\} \ni \sigma.$$

Theorem (Cyr-Kra-P)

If X is irreducible and LS, then $Aut(X,\sigma)$ is a LEF group

Restrictions on LS subshifts

$$\mathrm{Aut}(X,\sigma) = \{\phi\colon X \to X; \phi\circ\sigma = \sigma\circ\phi\} \ni \sigma.$$

Theorem (Cyr-Kra-P)

If X is irreducible and LS, then $Aut(X,\sigma)$ is a LEF group

Gordon-Vershik

The group G is Locally Embeddable into Finite groups (LEF) if for every finite set $K \subset G$, there exists a finite group H and a map $\varphi\colon G \to H$ such that the following hold:

1. $\varphi(k_1k_2) = \varphi(k_1)\varphi(k_2)$ for all $k_1, k_2 \in K$
2. the restriction of φ to K is injective.

LEF	not LEF
$\mathbb{Z}^d, \mathbb{F}_d, \mathbb{Q}$	$\langle a, b; ba^nb^{-1} = a^m\rangle$ $n > m \geq 2$
resid. finite	Thompson group $V\&T$

Restrictions on LS subshifts

$$\mathrm{Aut}(X,\sigma) = \{\phi\colon X \to X; \phi\circ\sigma = \sigma\circ\phi\} \ni \sigma.$$

Theorem (Cyr-Kra-P)

If X is irreducible and LS, then $Aut(X,\sigma)$ is a LEF group

There exists subshifts where $\mathrm{Aut}(X,\sigma)$ is not LEF.