Distribution asymptotique de la marche aléatoire de l'éléphant en régime superdiffusif

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À partir de travaux en collaboration avec Lucile Laulin[*],
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Rencontres Mathématiques de Rouen – Modèles aléatoires discrets
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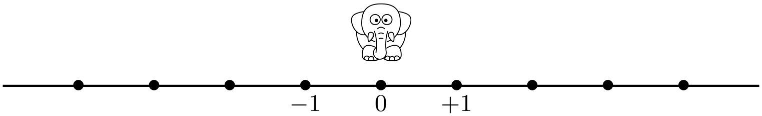
Model introduced by physics researchers, Schütz and Trimper (2004), in order to investigate the long-term memory effects in non-Markovian random walks.



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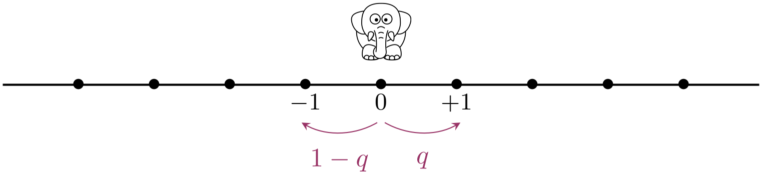

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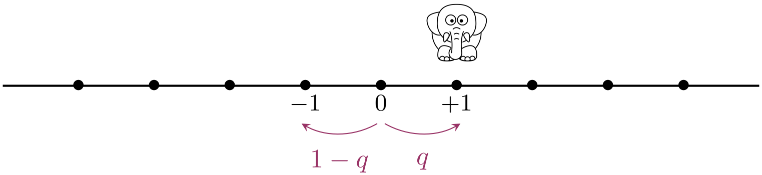

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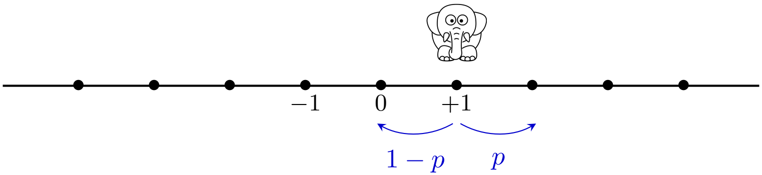

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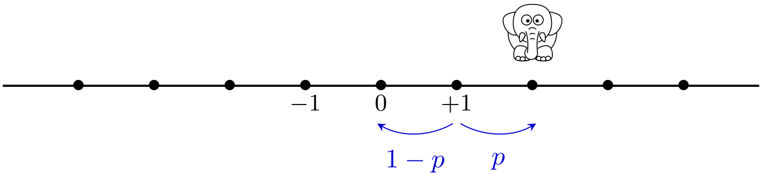

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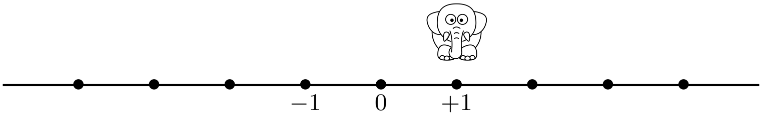

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Outline

2 - Asymptotics of the ERW

It is a specific case of step-reinforced RW. The position of the ERW at time n is given by $S_n = S_{n-1} + X_n$.
Let $a = 2p - 1$. We note that $a > \frac{1}{2} \Leftrightarrow p > \frac{3}{4}$.

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- Diffusive regime $(a < \frac{1}{2})$ [Baur-Bertoin (2016), Coletti et al. (2017)]

$$\frac{S_n}{n} \xrightarrow{a.s.} 0 \quad \text{and} \quad \left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_{t \geqslant 0} \Longrightarrow (W_t)_{t \geqslant 0}$$

where $(W_t)_{t \geqslant 0}$ is a centered Gaussian process with $\mathbb{E}[W_s W_t] = \frac{1}{1-2a} s \left(\frac{t}{s}\right)^a$.

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- Critical regime $(a = \frac{1}{2})$ [Bercu (2018), Coletti et al. (2017)]

$$\frac{S_n}{\sqrt{n} \log n} \xrightarrow{a.s.} 0 \quad \text{and} \quad \left(\frac{S_{\lfloor nt \rfloor}}{\sqrt{n^t \log n}}\right)_{t \geqslant 0} \Longrightarrow (B_t)_{t \geqslant 0}$$

  where $(B_t)_{t \geqslant 0}$ is a standard Brownian motion.

- Superdiffusive regime $(a > \frac{1}{2})$ [Baur-Bertoin (2016), Bercu (2018), Kubota-Takei (2019)]

$$\frac{S_n}{n^a} \xrightarrow{a.s.} L, \qquad \left(\frac{S_{\lfloor nt \rfloor}}{n^a}\right)_{t \geqslant 0} \Longrightarrow (t^a L)_{t \geqslant 0}$$

$$\text{and} \qquad \frac{S_n - n^a L}{\sqrt{n}} \xrightarrow{(d)} \mathcal{N}\left(0, \frac{1}{2a-1}\right)$$

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Those results have been extended to any dimension d taking $a = \frac{2dp-1}{2d-1}$ [Laulin (2019)].

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From Baur and Bertoin (2015) on Random Recursive Trees, we obtain

$$L = Z_1 C_1 + \sum_{i=2}^{\infty} (\beta_{\tau_i})^a Z_i C_i \quad \text{a.s.},$$

with Z_1 Rademacher r.v. $\mathcal{R}(q)$, $(Z_i)_{i \geqslant 2}$ Rademarcher $\mathcal{R}(1/2)$ r.v., C_i Mittag-Leffler r.v. with parameter a, β_k denotes a beta random variable with parameters $(1, k-1)$, and $\tau_i - 1$ Binomiale Negative $(i-1, 1-a)$ r.v.

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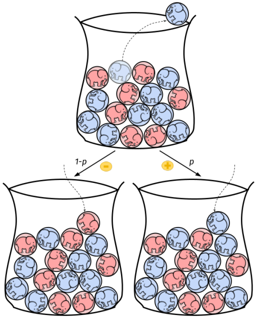
We denote by L_+ when $X_1 = +1$ and L_- when $X_1 = -1$.

The r.v. L is the discrete mixture of L_+ and L_-.

3 - Pólya urn representation of the ERW

Let $U_n = (R_n, B_n)$ with $\begin{array}{l} R_n = \#\text{ red balls} \\ B_n = \#\text{ blue balls} \end{array}$ at time n.

Initial composition of the urn is $\begin{array}{l} U_1 = (1, 0) \text{ with probability } q \\ U_1 = (0, 1) \text{ with probability } 1 - q \end{array}$. At each time n, a ball is drawn uniformly,



Source: L. Laulin.

Then, an extra ball of the same color is added with probability p, and of the other color with probability $1 - p$.

Let $S_1 = R_1 - B_1$, and for every $n \geqslant 1$

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We easily see that $(S_n)_{n \geqslant 1}$ is an ERW with a memory parameter p and first step parameter q.

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Let us recall that L_+ when $X_1 = +1$ and L_- when $X_1 = -1$.

Theorem (Janson (2004))

When $a > 1/2$,
$$U_1 = (1, 0), \lim_{n \to \infty} \frac{R_n - B_n}{n^a} = L_+ \ a.s.$$
$$U_1 = (0, 1), \lim_{n \to \infty} \frac{R_n - B_n}{n^a} = L_- \ a.s.$$

By symmetry, $L_- = -L_+$.

We prove that L_+ satisfies a fixed-point equation, which will be the key to studying its distribution.

To this aim, we associate a tree structure to the urn process: the tree grows at each drawing from the urn.

At time 1, the tree is one red node with proba q or one blue node with proba $1-q$.

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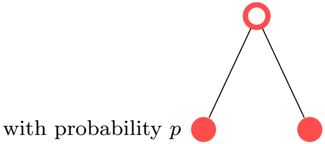

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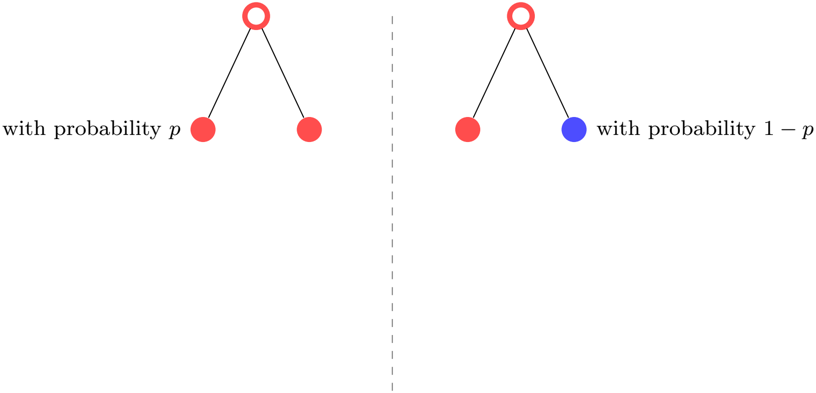

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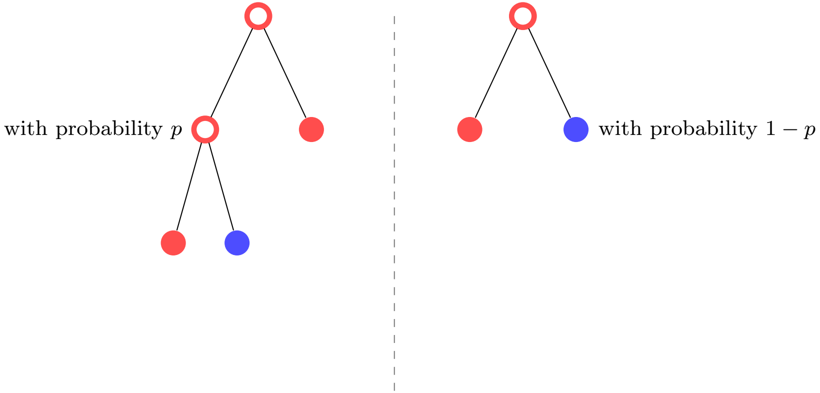

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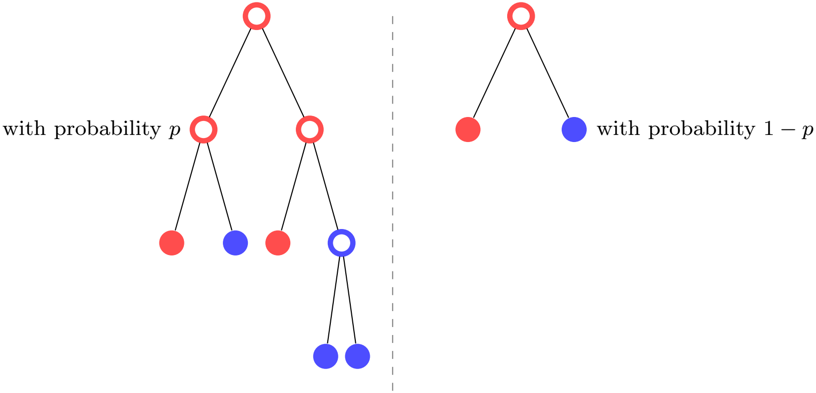

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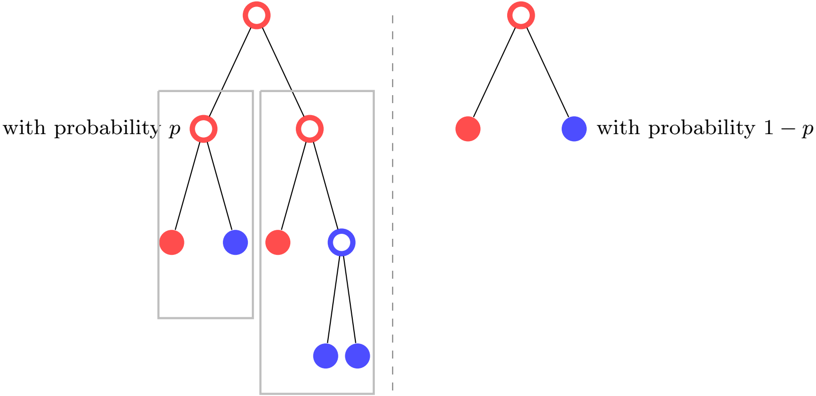

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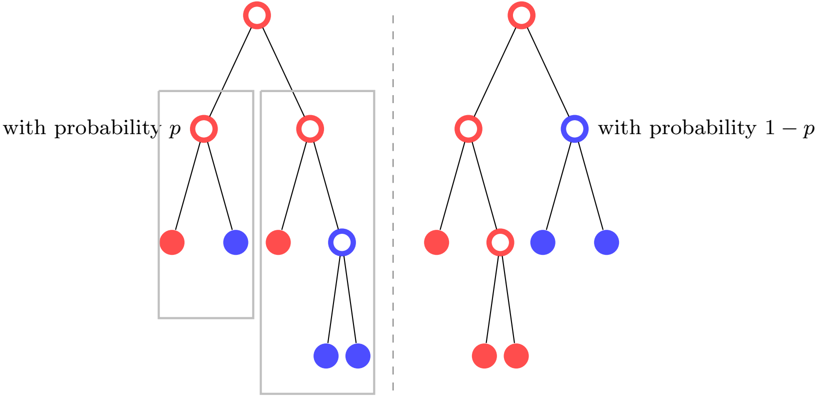

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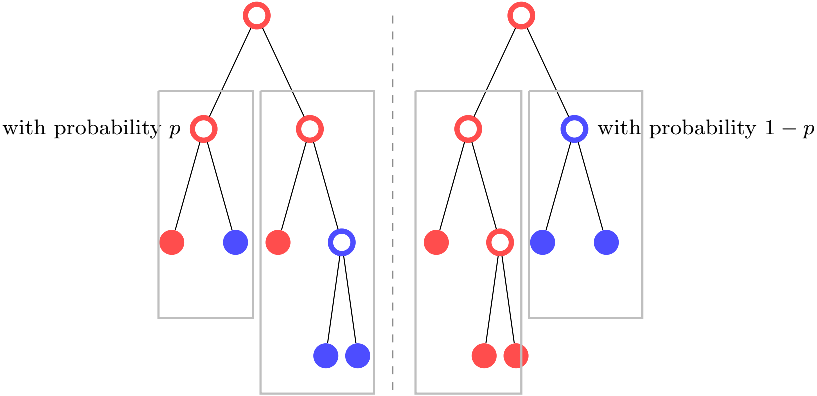

At time n, each leaf in the tree represents a ball in the urn.

For $n \geqslant 2$, $D_1(n)$ be the number of leaves at time n of the first subtree
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We have $D_1(n) + D_2(n) = n$, and

$$U_{(1,0)}(n) \stackrel{\mathcal{L}}{=} U^{(1)}_{(1,0)}(D_1(n) - 1) + \xi_p U^{(2)}_{(1,0)}(D_2(n) - 1) + (1 - \xi_p) U^{(2)}_{(0,1)}(D_2(n) - 1).$$

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We recall that, when

$$U_1 = (1, 0), \lim_{n\to\infty} \frac{R_n - B_n}{n^a} = L_+ \text{ a.s.}$$

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$$\lim_{n\to+\infty} \frac{D_1(n)}{n} = V \qquad \lim_{n\to+\infty} \frac{D_2(n)}{n} = 1 - V \quad \text{with } V \sim \mathcal{U}nif[0, 1].$$

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Let $n_1 = D_1(n) - 1$ and $n_2 = D_2(n) - 1$, then

$$\frac{U_{(1,0)}(n)}{n^a} \overset{\mathcal{L}}{=} \left(\frac{n_1}{n}\right)^a \frac{U^{(1)}_{(1,0)}(n_1)}{n_1^a} + \xi_p \left(\frac{n_2}{n}\right)^a \frac{U^{(2)}_{(1,0)}(n_2)}{n_2^a} + (1 - \xi_p) \left(\frac{n_2}{n}\right)^a \frac{U^{(2)}_{(0,1)}(n_2)}{n_2^a}.$$

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$$\frac{U_{(1,0)}(n)}{n^a} \stackrel{\mathcal{L}}{=} \left(\frac{n_1}{n}\right)^a \frac{U^{(1)}_{(1,0)}(n_1)}{n_1^a} + \xi_p \left(\frac{n_2}{n}\right)^a \frac{U^{(2)}_{(1,0)}(n_2)}{n_2^a} + (1-\xi_p)\left(\frac{n_2}{n}\right)^a \frac{U^{(2)}_{(0,1)}(n_2)}{n_2^a}.$$

Taking the limit,

Theorem (G., Laulin, Raschel, 2023)

L_+ satisfies the fixed-point equation

$$L_+ \stackrel{\mathcal{L}}{=} V^a L_+^{(1)} + \xi_p (1-V)^a L_+^{(2)} + (1-\xi_p)(1-V)^a L_-^{(2)}$$

$$\Longrightarrow \qquad L_+ \stackrel{\mathcal{L}}{=} V^a L_+^{(1)} + (2\xi_p - 1)(1-V)^a L_+^{(2)} \qquad (\textit{because } L_- = -L_+)$$

where all the r.v. are independent and

- V *uniform random variable on* $[0,1]$,
- ξ_p *Bernoulli random variable with parameter* p,
- $L_+^{(1)}$ *and* $L_+^{(2)}$ *are copies of* L_+.

Main results

$$L_+ \stackrel{\mathcal{L}}{=} V^a L_+^{(1)} + (2\xi_p - 1)(1-V)^a L_+^{(2)}.$$

We deduce from this fixed-point equation that

Theorem (G., Laulin, Raschel, 2023)

- $\mathrm{Supp}(L_+) = \mathbb{R}$.
- L_+ *has a bounded smooth density on* $\mathbb{R}$ (based on the study of the characteristic function).
- *The moments of* L_+ *characterize the distribution.*
- $\mathbb{E}\big[\exp(L_+^2)\big] < +\infty$.

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Theorem (G., Laulin, Raschel, 2023)

- $\mathrm{Supp}(L) = \mathbb{R}$.
- *L has a bounded smooth density on $\mathbb{R}$.*
- *The moments of L characterize the distribution.*
- $\mathbb{E}\big[\exp(L^2)\big] < +\infty$.

Because L is a discrete mixture of L_+ and L_-, and $L_- = -L_+$.

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These results are still true in higher dimension.

Focus on the density function

From the fixed-point equation, we deduce a nice recursive way to compute de successive moments of L_+.

Theorem (G., Laulin, Raschel, 2023)

The moments of L_+ *are given by the following recursive equation. Let* $(m_k)_{k\geqslant 1}$ *with* $m_1 = 1$*, and for* $k \geqslant 2$*,*

$$m_k = \frac{1}{ka - c_k} \sum_{j=1}^{k-1} c_j m_j m_{k-j},$$

where $c_k = 1$ *for even* k *and* $c_k = a$ *for odd* k*. Let* C *be a Mittag-Leffler r.v. with parameter* a*.*
Then for any $k \geqslant 1$*,*

$$\mathbb{E}[L_+^k] = \mathbb{E}[C^k] m_k \quad \textit{with } \mathbb{E}[C^k] = \frac{k!}{\Gamma(ka+1)}.$$

Warning! We have for any $k \geqslant 1$, $\mathbb{E}[L_+^k] = \mathbb{E}[C^k] m_k$ with C a Mittag-Leffler variable, but we cannot write

$$L_+ = C.Y \quad \text{with } Y \text{ independent of } C \text{ and } \mathbb{E}[Y^k] = m_k.$$

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~~$L_+ = C.Y$ with Y an independent variable of C and $\mathbb{E}[Y^k] = m_k$.~~

However, we can study the sequence $(m_k)_{k\geqslant 1}$ to obtain information on the density of L_+.

Theorem (G., Laulin, Raschel, Simon, 2024)

We have

$$m_k \underset{k\to+\infty}{\sim} \frac{2a}{a+1}\rho(a)^k,$$

where $\rho(a) > 1$ is an explicit constant.

From the properties of the Mittag-Leffer distribution, we deduce

$$\frac{\mathbb{E}[L_+^k]}{\rho(a)^k k!} \underset{k\to+\infty}{\sim} \frac{2a}{(a+1)\Gamma(1+ak)}.$$

By Kasahara's Tauberien theorem, we deduce

Theorem (G., Laulin, Raschel, Simon, 2024)

Let f be the density function of L_+. We have

$$\ln f(x) \underset{|x|\to\infty}{\sim} -(1-a)\left(\frac{a^a}{\rho(a)}x\right)^{\frac{1}{1-a}}.$$

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Note that for $a > 1/2$, $\frac{1}{1-a} > 2$. The distribution is clearly sub-Gaussian.

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A more precise result: *there are explicit positive constants $c_+(a), c_-(a)$ such that*

$$f(x) \underset{x\to+\infty}{\sim} c_+(a)x^{\frac{2a-1}{2(1-a)}}\,\mathrm{e}^{-(1-a)\left(\frac{a^a}{\rho(a)}x\right)^{\frac{1}{1-a}}}$$

$$f(x) \underset{x\to-\infty}{\sim} c_-(a)x^{\frac{2a^2-3a-1}{2(1-a^2)}}\,\mathrm{e}^{-(1-a)\left(\frac{a^a}{\rho(a)}x\right)^{\frac{1}{1-a}}}.$$

By Kasahara's Tauberien theorem, we deduce

Theorem (G., Laulin, Raschel, Simon, 2024)

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Note that for $a \in (0,1)$, $\frac{2a-1}{2(1-a)} > \frac{2a^2-3a-1}{2(1-a^2)}$.

Using the following recursive expression of the distribution of S_n

$$Q(n+1,k) = (np - ak)Q(n,k) + ((1-p)n + a(k-1))Q(n,k-1), \quad Q(1,1) = 1,$$

with $Q(n,k) = (n-1)!\mathbb{P}(S_n = 2k-n)$, we also proved

- the distribution of L_+ is unimodal;
- the density est log-concave for $a \in (1/2, a_0)$ for an explicit $a_0 < 1$.

Summary and some open questions

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- It was known that the normalized ERW converges to a subgaussian nondegenerate random variable L when $a = 2p - 1 > \frac{1}{2}$ $(p > \frac{3}{4})$.
- Nothing was known about the asymptotic distribution, except that it has no atoms.
- Using a tree decomposition of the Pólya urn representation of ERW, we have been able to prove that L_+ satisfies a fixed-point equation in distribution.
- From this fixed-point equation, we deduced that L_+ has a smooth bounded positive density on $\mathbb{R}$, the distribution is characterized by its moments.
- From the recursive equation on the moments, we obtained the tails of the distribution.

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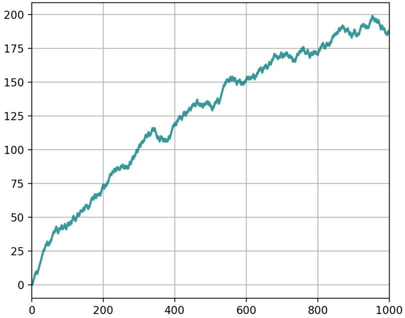
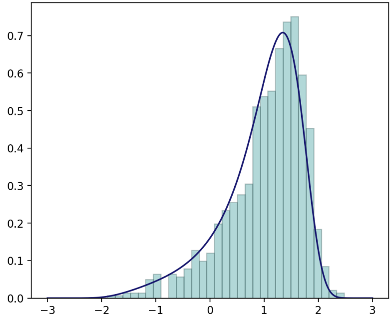
Natural open questions:

- Is the density log-concave for $a \in (1/2, 1)$?
- The ERW is a specific step-reinforced random walk. Would it be possible to extend some of these results to this more general class of non-makovian random walks?

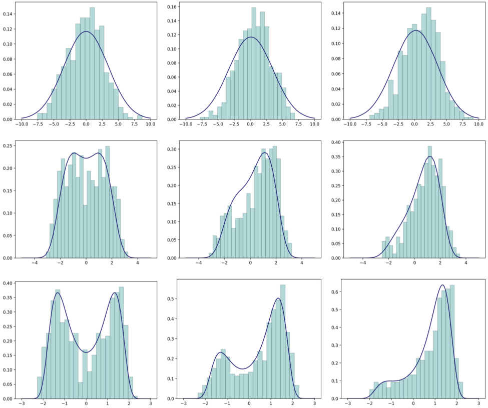


A trajectory of S_n, with $q = 1$ and $p = 0.92$.



Histogram and moments approximation of the density of L_+.

Thank you!



$q = 0.5$ $q = 0.7$ $q = 0.9$

Histogram and moments approximation of the density of L_q.

First (resp. second, third) line: $p = 0.77$ (resp. $p = 0.87$, $p = 0.92$).