Forêts massiques : distribution exacte et limite locale sur le graphe complet

MAD days 2025, LMRS, Rouen

20 juin 2025, 14h45 (heure de Paris)

Matteo D'Achille

basé sur

- D'A, Nathanaël Enriquez (Orsay & ENS), Paul Melotti (Orsay) 2403.11740
- D'A, Wioletta Ruszel (Utrecht) WIP

λ massive spanning forest: definition

A simple graph G (no self-loops), the set of its **spanning forests** with k components $\mathcal{F}_k(G)$, $k \geq 1$ ($\mathcal{F}_1(G)$: spanning trees of G).

Def. ($\lambda\mathbf{SF}(G)$). For $f \in \mathcal{F}_k(G)$, let:

- $C(f)$: set of connected components (trees);
- For a tree t, let $|t| = \#\{\text{vertices in } t\}$.

Then for $\lambda > 0$ define

$$\mathbf{P}\left(\lambda\mathbf{SF}(G) = f\right) \propto \lambda^{|C(f)|} \prod_{t \in C(f)} |t| \, .$$

Motivations and related works

Massive (off-critical) version of $\mathbf{UST}(G)$.

$$\lambda\mathbf{SF}(G) \xrightarrow[\lambda\downarrow 0]{\text{(d)}} \mathbf{UST}(G)$$

- Grimmett 1980: cardinality of h-th progeny of a given vertex in $\mathbf{UST}(\mathbf{K}_n)$
- Nachmias–Peres 22: put Grimmett into a local limit results + extension to $\mathbf{UST}(G_n)$, $\{G_n\}_{n\geq 1}$ a seq. of simple, connected regular graphs with diverging degree.

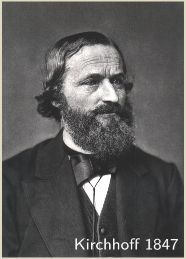
Some confusion around Kirchhoff's thm(s).

Kirchhoff's matrix-forest theorem

Biggs 93, Kenyon 11

For $\Delta_G = D_G - A_G$ the graph laplacian, define the **characteristic polynomial** of G:

$$P_G(\lambda) \stackrel{\text{def}}{=} \det(\Delta_G + \lambda I_n)$$



Kirchhoff's matrix-forest theorem:

$P_G(\lambda)$ is the **OGF** of $\{\mathcal{F}_k^\bullet(G)\}_{k\geq 1}$

$P_G(\lambda) = \sum_{k\geq 1} |\mathcal{F}_k^\bullet(G)|\lambda^k$,

$\mathcal{F}_k^\bullet(G)$: **rooted** spanning forests with k trees.

$\lambda\mathrm{SF}(G)$ is determinantal

Burton–Pemantle 93 for **UST**(G). Let $n = |V(G)|$.

$R_\lambda = (\Delta_G + \lambda I_n)^{-1}$ **Green function**

$K_\lambda(e,f) = R_\lambda(e_-, f_-) + R_\lambda(e_+, f_+) - R_\lambda(e_-, f_+) - R_\lambda(e_+, f_-)$ **transfer current matrix** (given an orientation)

Proposition (D'A–Enriquez–Melotti 24)

Let $e_1, \ldots, e_k, e_{k+1}, \ldots, e_p$ be p distinct edges of G. Then

$$\mathbf{P}\left(e_1, \ldots, e_k \in \lambda\mathbf{SF}(G),\ e_{k+1}, \ldots, e_p \notin \lambda\mathbf{SF}(G)\right) = \det M$$

where

$$M_{i,j} = \begin{cases} K_\lambda(e_i, e_j) & \text{if } i \leq k \\ \delta_{i,j} - K_\lambda(e_i, e_j) & \text{if } k < i \leq p. \end{cases}$$

$\lambda\mathbf{SF}(G)$ is determinantal

Proof. Start with "all out" ($k = 0$):

$$\mathbf{P}\left(e_1, \ldots, e_p \notin \lambda\mathbf{SF}(G)\right) = \frac{P_{G\setminus\{e_1,\ldots,e_p\}}(\lambda)}{P_G(\lambda)} = \frac{\det\left(\Delta_{G\setminus\{e_1,\ldots,e_p\}} + \lambda I_n\right)}{\det\left(\Delta_G + \lambda I_n\right)}$$

Observe that

$$\Delta_{G\setminus\{e_1,\ldots,e_p\}} = \Delta_G - BB^T$$

for an easy matrix B. Use now

$$\det\left(I_n - R_\lambda BB^T\right) = \det\left(I_p - B^T R_\lambda B\right)^*$$

$B^T R_\lambda B = K_\lambda$. The case $k \geq 1$ by inclusion-exclusion. ■

*Sylvester's determinant theorem for rectangular matrices

Transfer current matrix for $\mathbf{K}_n$

(D'A–Enriquez–Melotti 24)

From now on: $G = \mathbf{K}_n$, where $\mathbf{K}_n$ is the **complete graph** on n vertices.

$\Delta_{\mathbf{K}_n}$ is 2-valued, so R_λ is. Compute these two values and plug in: $\forall e, f \in E(\mathbf{K}_n)$,

$$K_\lambda(e, f) = \frac{1}{(n+\lambda)}\left(\mathbf{1}_{e_-=f_-} + \mathbf{1}_{e_+=f_+} - \mathbf{1}_{e_-=f_+} - \mathbf{1}_{e_+=f_-}\right) .$$

Remark: e, f have to be incident: $+1$ if they start or end at the same vertex, -1 if they "chase" each other, otherwise 0!

The key lemma

t_{lab}: tree w. vertices **labeled** by distinct integers in $\{1,\ldots,n\}$.
Let $k = |t_{\mathrm{lab}}|$.

Lemma [Inclusion proba] (D'A–Enriquez–Melotti 24)

$$\mathbf{P}\left(t_{\mathrm{lab}} \subset \lambda\mathbf{SF}(\mathbf{K}_n)\right) = \frac{k}{(n+\lambda)^{k-1}} \, .$$

Proof.

$$M_\lambda = \frac{1}{n+\lambda}\mathcal{A}\,\mathcal{A}^T \, ,$$

$\mathcal{A}$: $(k-1)\times k$ edge-vertex incidence matrix. By **Cauchy–Binet**:

$$\det M_\lambda = \frac{1}{(n+\lambda)^{k-1}} \sum_{j=1}^{k} (\det \mathcal{A}_j)^2$$

■

Local limit of $\lambda\mathbf{SF}(\mathbf{K}_n)$

Main theorem

Let $\mathbf{o}$ be a fixed vertex of $\mathbf{K}_n$ (viewpoint). Let:

- $T_{n,\lambda}$ labeled tree of $\mathbf{o}$ in $\lambda\mathbf{SF}(\mathbf{K}_n)$;
- Shape$(T_{n,\lambda}, \mathbf{o})$: non-planar unlab. tree obt. from $T_{n,\lambda}$, rooted at $\mathbf{o}$;
- **Local convergence**: endow locally finite (but possibly ∞) rooted trees with the topology inherited from the product topology

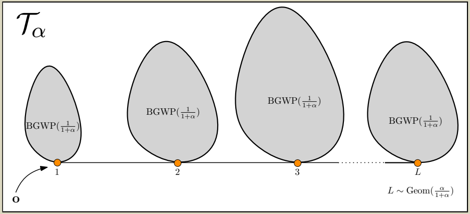
Local limit of $\lambda\mathrm{SF}(\mathbf{K}_n)$

Main theorem

Theorem (D'A–Enriquez–Melotti 24)

Let $\lambda = \lambda_n$. The following local convergence holds:

$$\mathrm{Shape}(T_{n,\lambda_n}, \mathbf{o}) \xrightarrow[n\to\infty]{(d)} \begin{cases} \mathcal{T}_0 & \text{if} \quad \lambda_n = o(n) \\ \mathcal{T}_\alpha & \text{if} \quad \lambda_n \sim \alpha n \\ \{\mathbf{o}\} & \text{if} \quad \lambda_n \gg n\,. \end{cases}$$



Local limit of $\lambda\mathbf{SF}(\mathbf{K}_n)$

Remarks:

- $\mathcal{T}_0 = \mathrm{BGWP}(1)$ condit. to survival (Grimmett, generalized by Nachmias–Peres);
- $\mathcal{T}_\alpha$ is **unimodular** (Benjamini 13);
- # of trees in $\lambda\mathbf{SF}(\mathbf{K}_n)$ concentrates around $\frac{\lambda+1}{\lambda+n}n$ (Castell–Avena–Gaudilliere–Melot);

☞ Exact distribution of Shape$(T_{n,\lambda}, \mathbf{o})$ at for any n and λ!

Exact distribution of Shape$(T_{n,\lambda}, \mathbf{o})$

Sketch of proof

Key lemma $\Rightarrow$ Proba that the (labeled) connected component of $\lambda\mathbf{SF}(\mathbf{K}_n)$ containing $\mathbf{o}$ = a given (labeled) tree is of binomial type!

$$\mathbf{P}\left(\mathrm{Shape}(T_{n,\lambda}, \mathbf{o})_{\leq h} = t\right) = \frac{1}{|\mathrm{Aut}(t)|} \frac{(n-1)!}{(n-|t|)!} \frac{1}{(n+\lambda)^{|t|}} (n|t_h| + \lambda|t|) \left(1 - \frac{|t_{<h}|}{n+\lambda}\right)^{n-|t|-1}.$$

t: a rooted, unlabeled tree and integer $h \geq 0$,

- t_h vertices of t at distance exactly h from the root;
- $t_{\leq h}$: tree given by the intersection of t with the closed ball of radius h centered at the root;
- $t_{<h}$: tree given by the intersection of t with the closed ball of radius $h-1$ centered at the root;
- Aut(t) graph automorphisms of t preserving its root.

☞ The phase transition for $\lambda = \lambda_n$ follows.

Identification of the limit

(Featuring a simple lemma on subcritical BGWP)

Lemma (D'A–Enriquez–Melotti 24)

$\beta < 1$, $h \geq 0$, t: an unlabeled rooted tree of height $\leq h$.

$$\mathbf{P}\left(\mathsf{BGWP}(\beta)_{\leq h} = t\right) = \frac{1}{|\mathsf{Aut}(t)|}\beta^{|t|-1}e^{-\beta|t_{<h}|} .$$

Proof. LHS and RHS satisfy the same recurrence with same initial condition. ■

Spinal decomposition and use the Lemma with $\beta = \frac{1}{1+\alpha}$.

Three perspectives on $\mathbb{Z}_d$

For $n_1 \ldots, n_d$ positive integers, $G_{n_1,n_2,\ldots,n_d}$: Cartesian product of $d \geq 1$ path graphs on $n_1 \ldots, n_d$ vertices.

- $\lambda\mathbf{SF}(G_{n_1,n_2})$ and colored partitions of n_2 avoiding the pattern $1^1 1^2$ (with Melotti)
- ☞ Local limit of $\lambda\mathbf{SF}(G_{n_1,n_2,\ldots,n_d})$ (with Ruszel)
- Exit probabilities and mSLE in the scaling limit

$\lambda\mathbf{SF}$ and fGFF

For UST($\mathbb{Z}_d$):

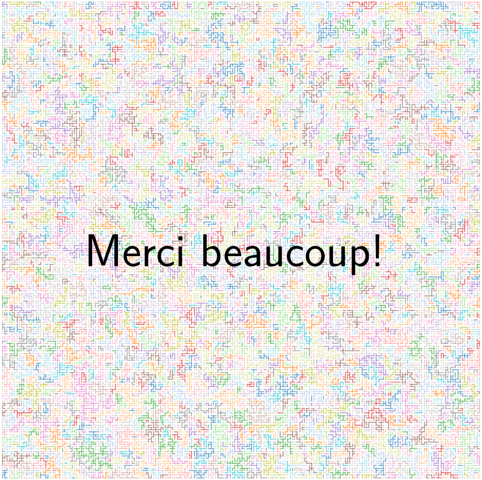
Local observables (degree field, ...) $\iff$ Expect. values of gradients of Grassmannian variables wrt fermionic discrete GFF (Chiarini–Cipriani–Rapoport–Ruszel 23)

This works also for $\lambda\mathbf{SF}(\mathbb{Z}_d)$!

Proposition (D'A–Ruszel 24^+)

$$\mathbf{P}\left(e \in \lambda\mathbf{SF}(\mathbb{Z}_d)\right) = \frac{1}{d} - f_d(\lambda)$$

for some explicit function $f_d(\lambda)$ s.t. $f_d(\lambda) \underset{\lambda\downarrow 0}{\to} 0$



Sample of $\lambda\mathbf{SF}(G_{200,200})$ at "large" λ