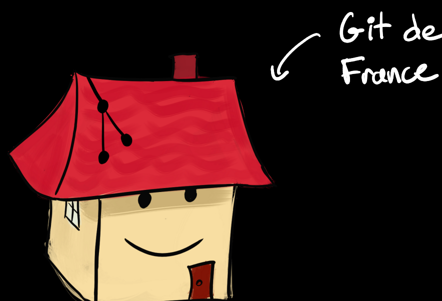
Which Distribution for Git Graphs

(ongoing work)

Julien COURTIEL (Université de Caen Normandie)
with Martin PEPIN (Université de Caen Normandie)

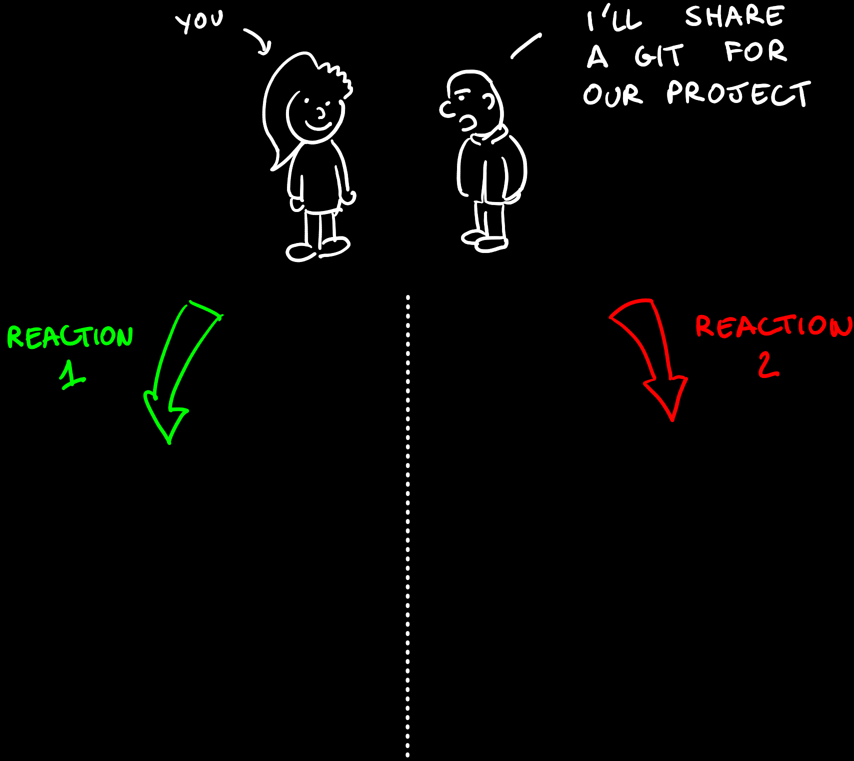


MAD Days 2025 (Rouen)

INTRODUCTION

HOW GIT WORKS

SITUATION PLAY



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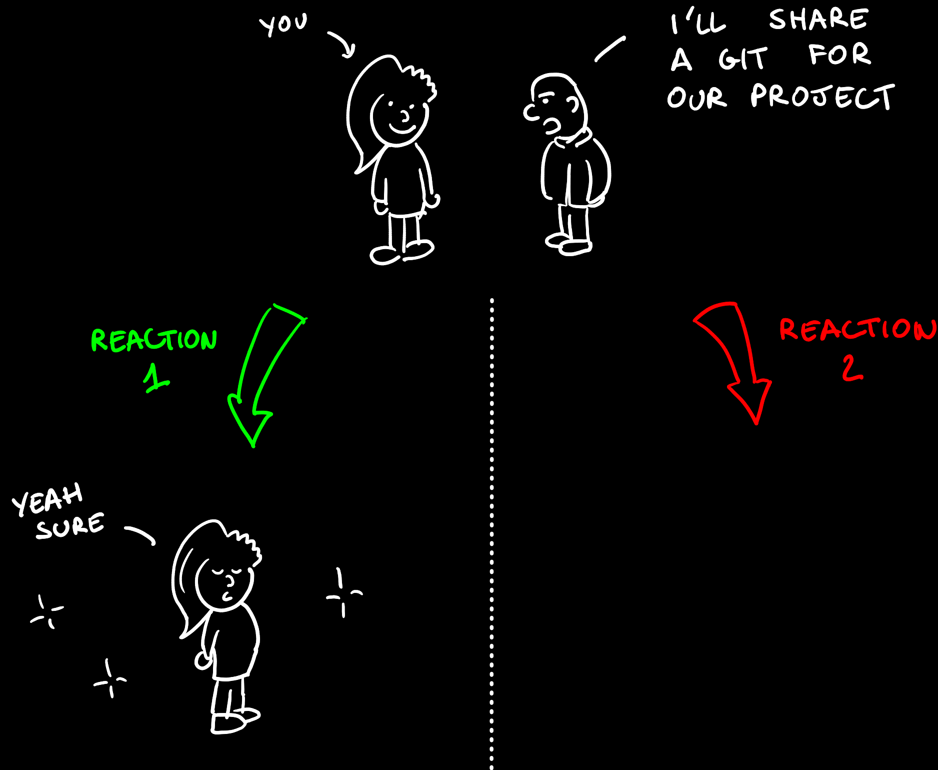

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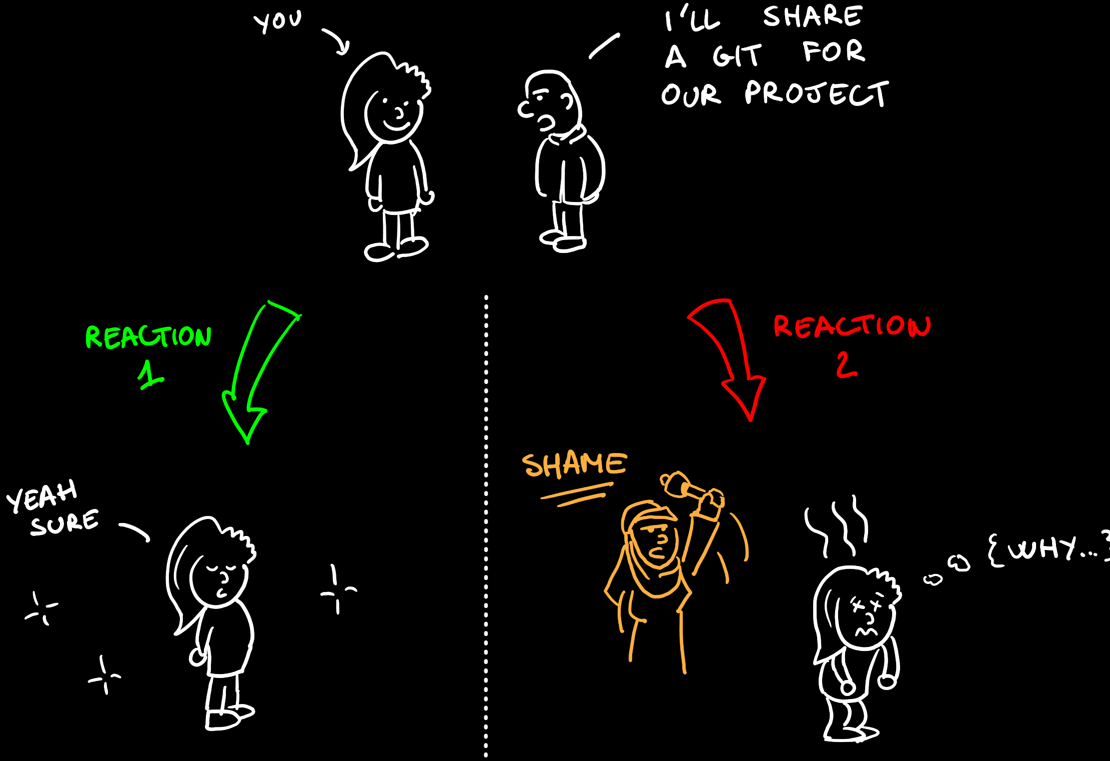

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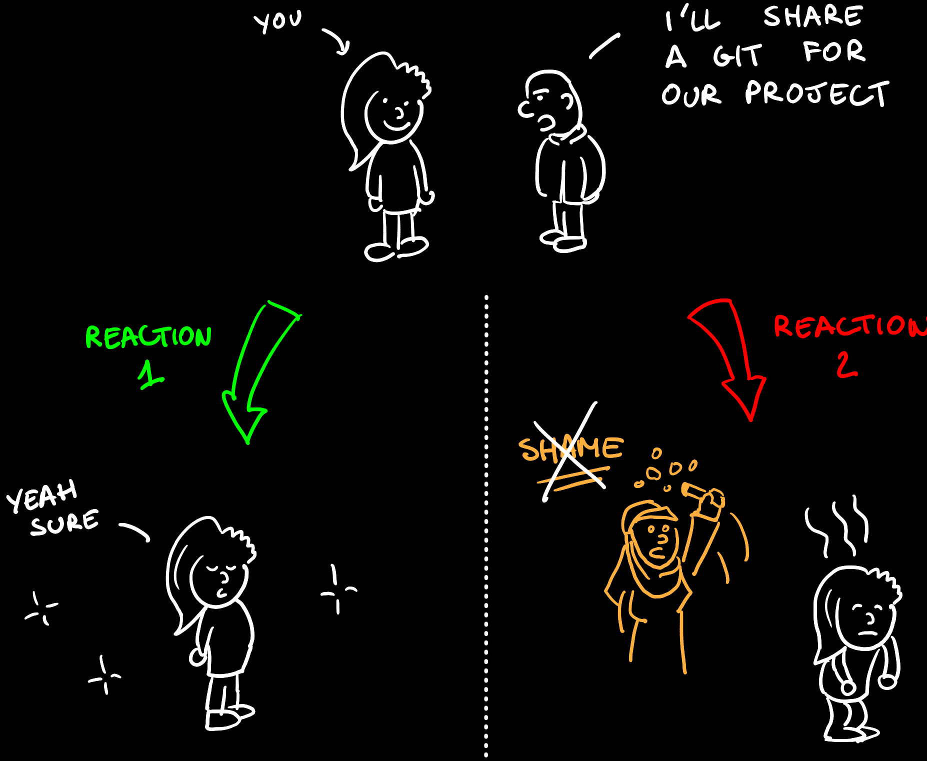

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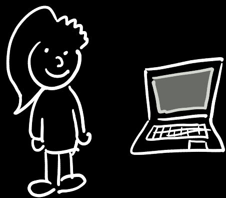

YOU

Situation play 2

You write an article
with an obscure coauthor

OBSCURE
COAUTHOR

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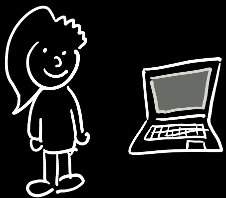

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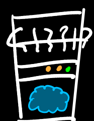

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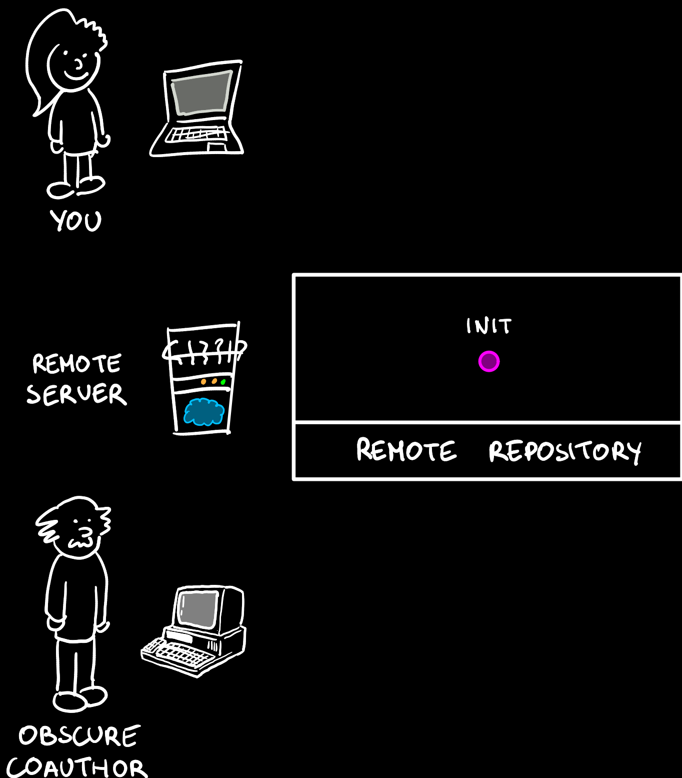

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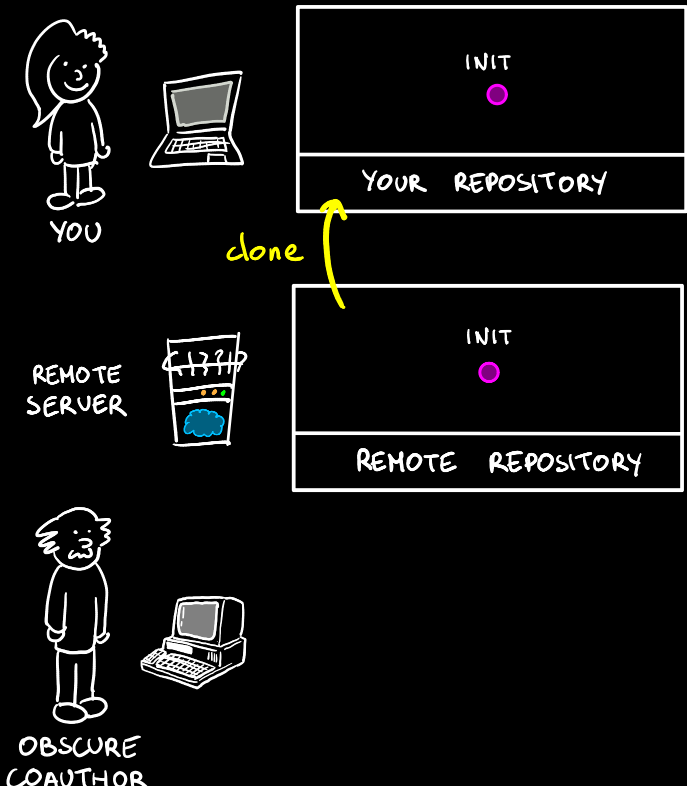

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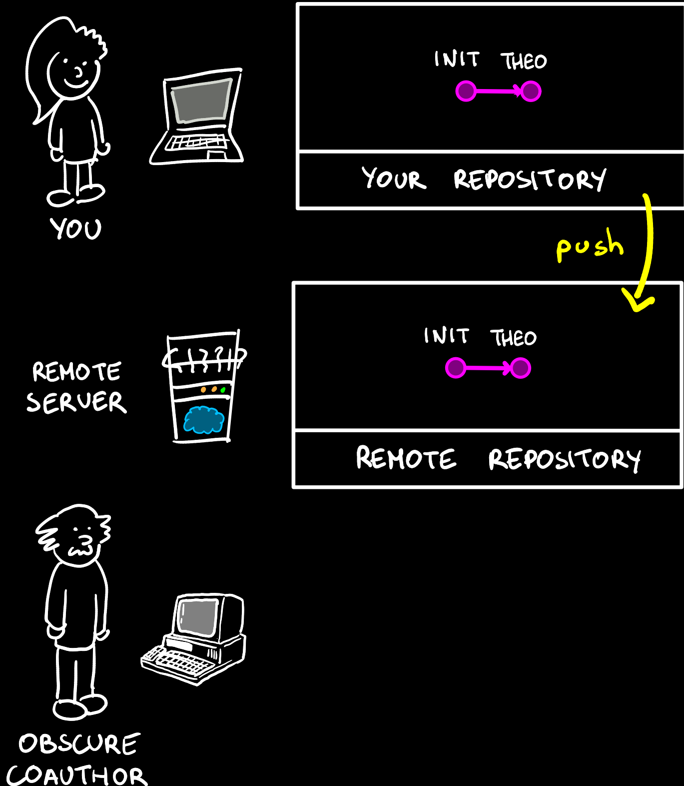

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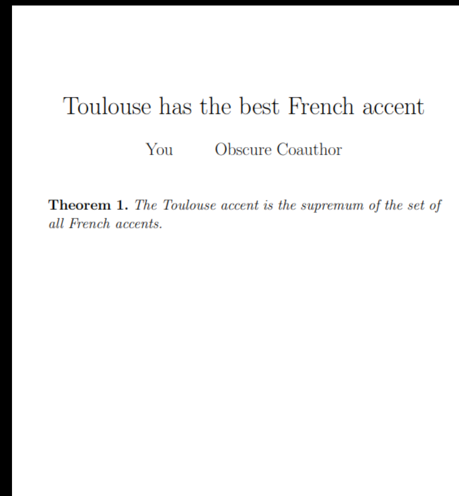

Toulouse has the best French accent

You Obscure Coauthor

Theorem 1. *The Toulouse accent is the supremum of the set of all French accents.*

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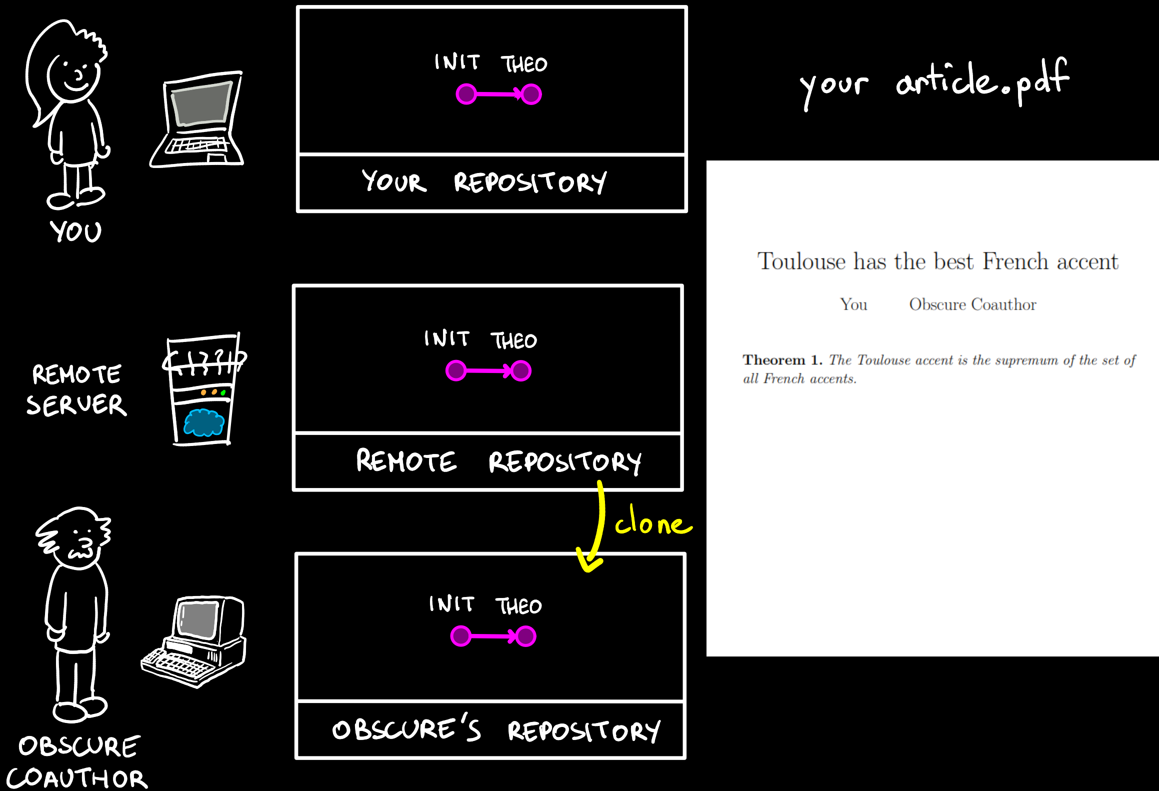

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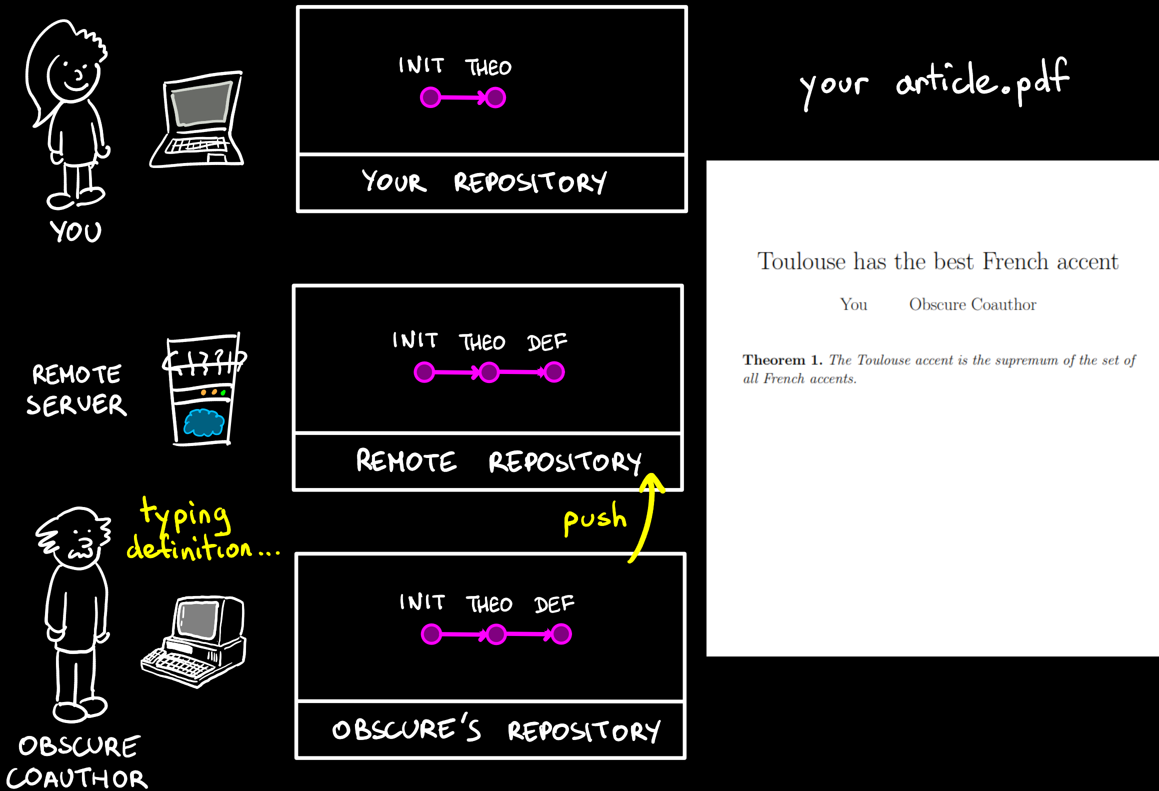

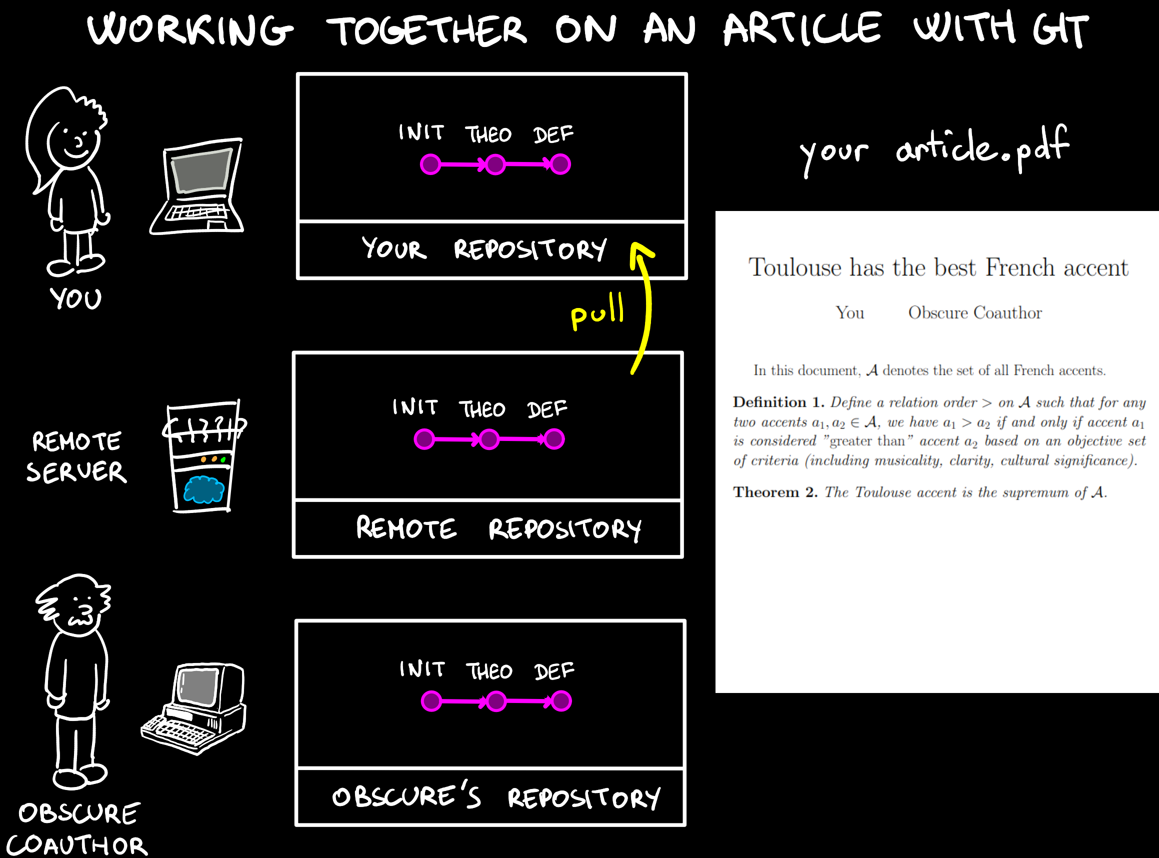
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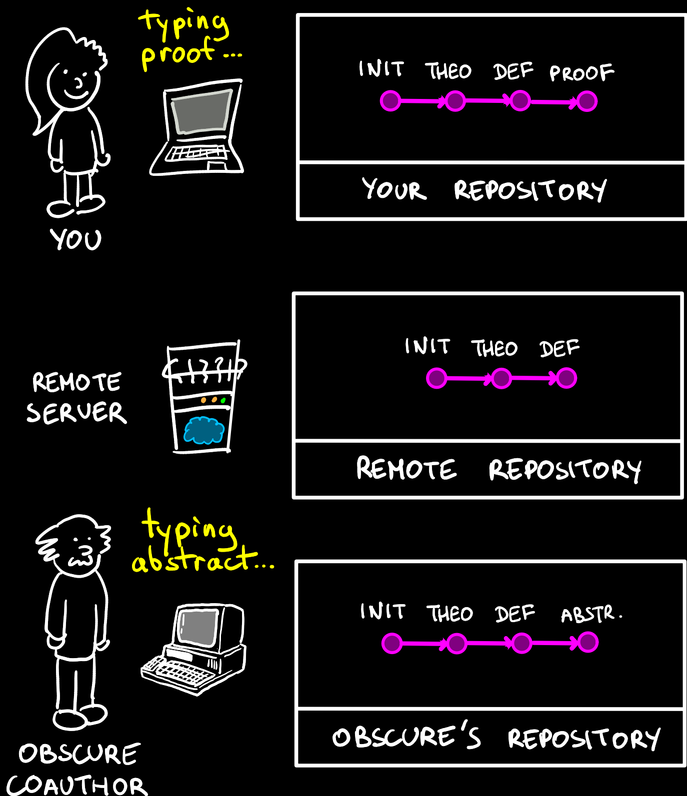

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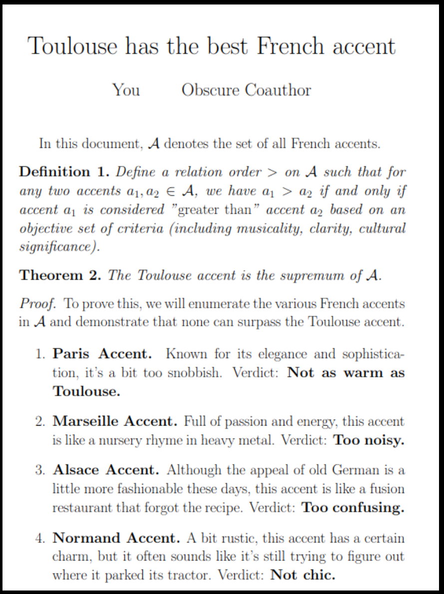

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3. **Alsace Accent.** Although the appeal of old German is a little more fashionable these days, this accent is like a fusion restaurant that forgot the recipe. Verdict: **Too confusing.**
4. **Normand Accent.** A bit rustic, this accent has a certain charm, but it often sounds like it's still trying to figure out where it parked its tractor. Verdict: **Not chic.**

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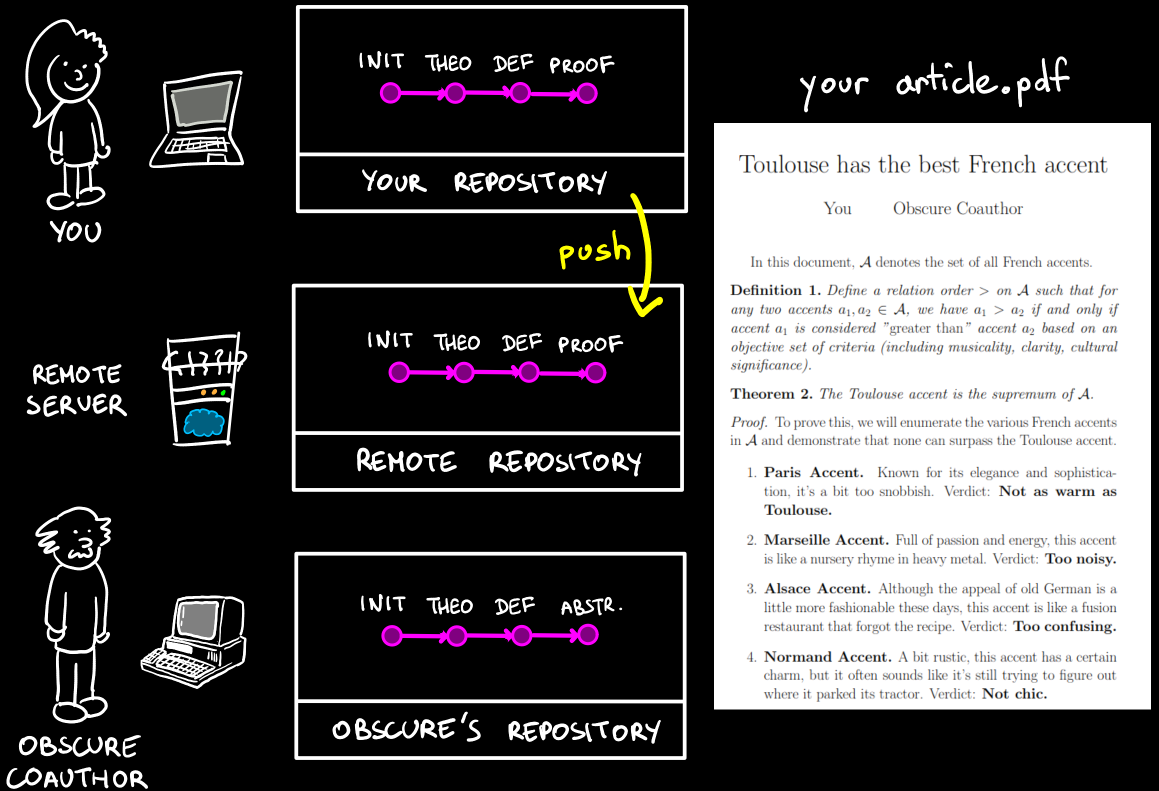

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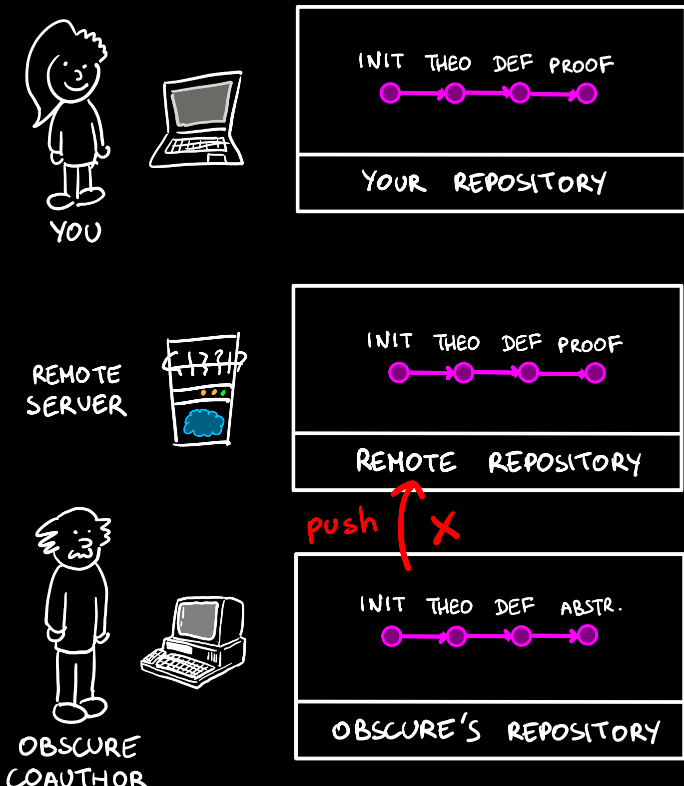

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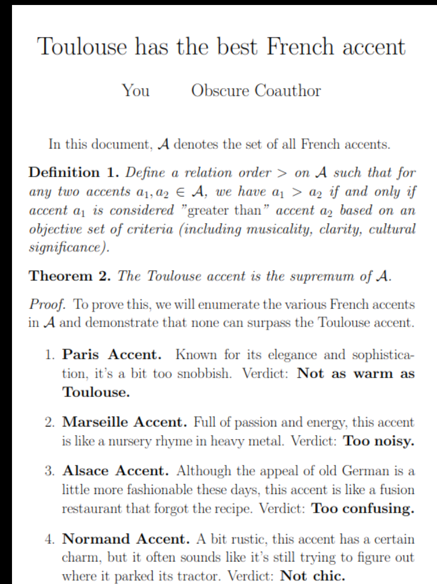

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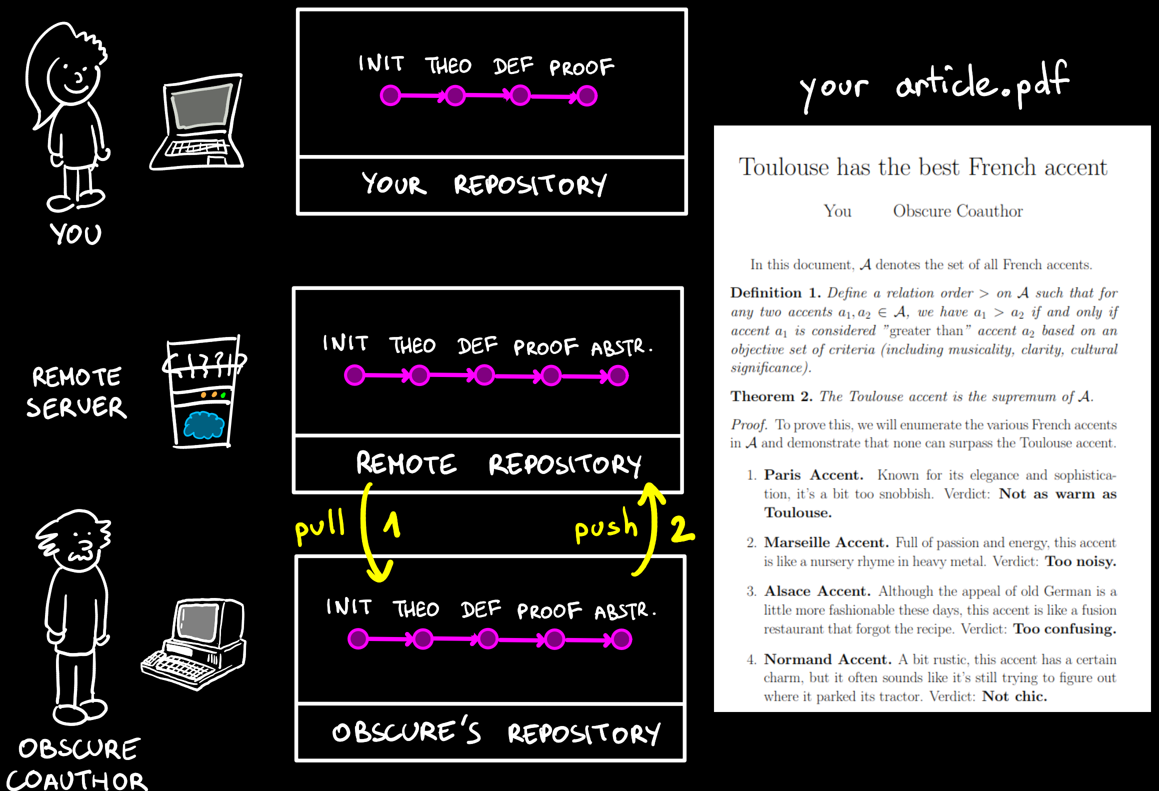

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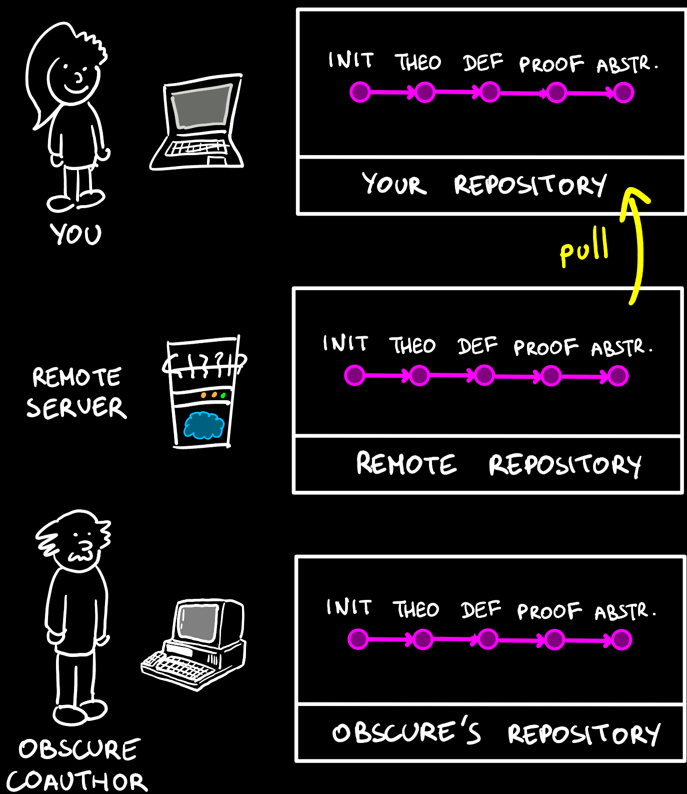

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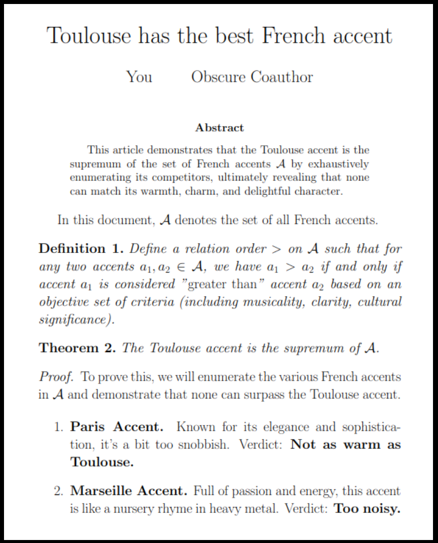

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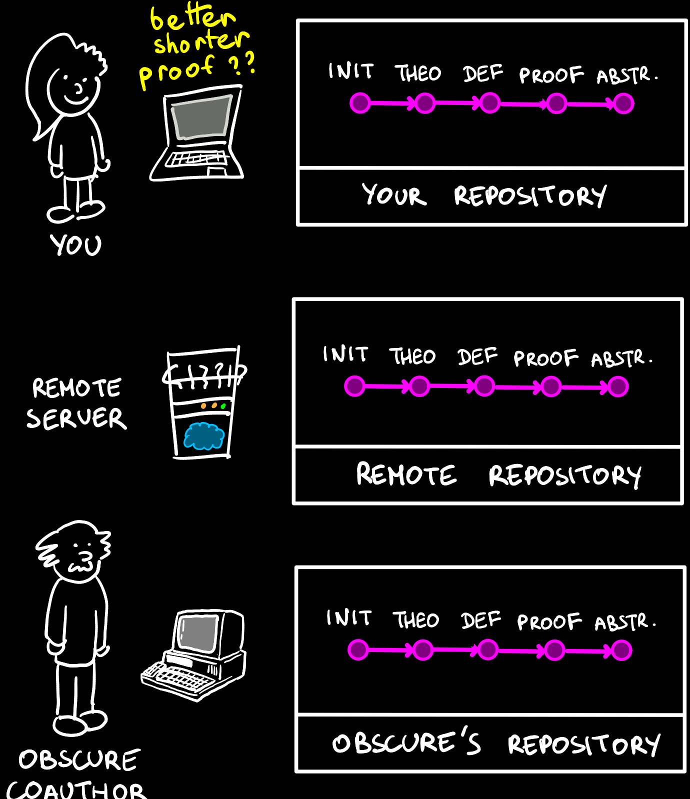

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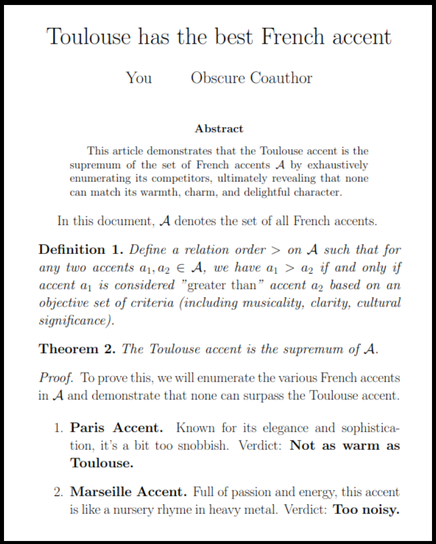

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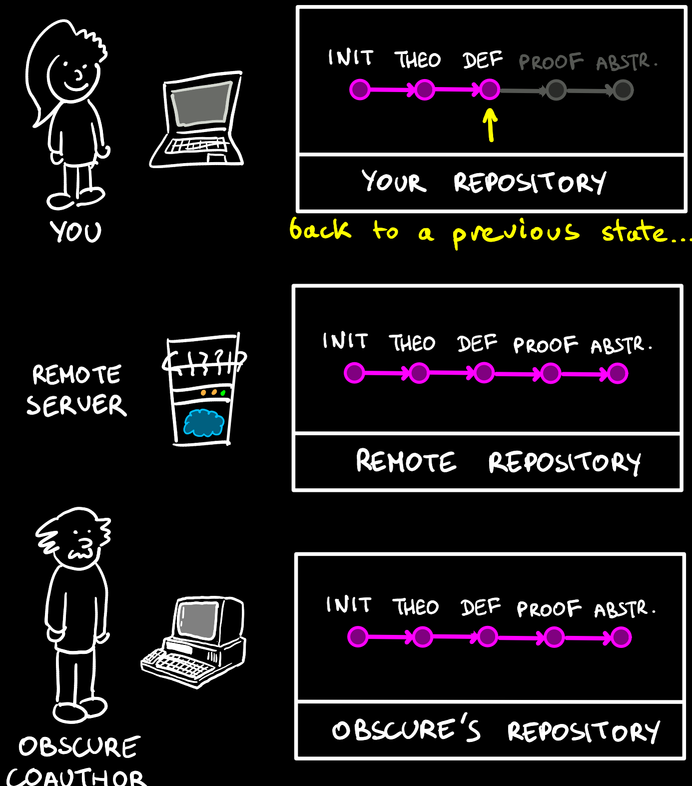

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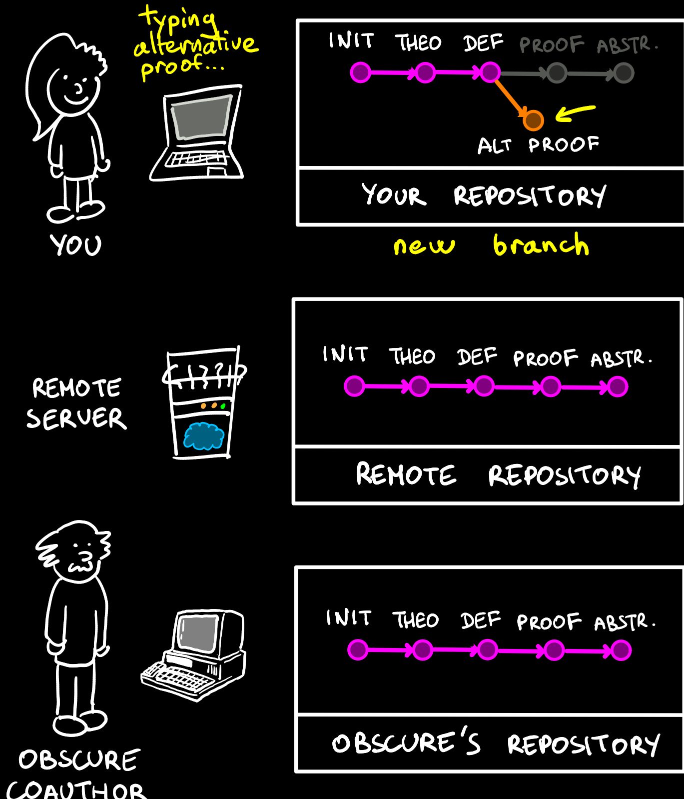

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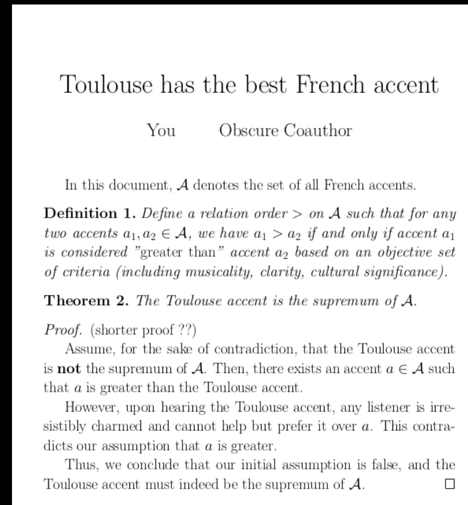

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Proof. (shorter proof ??)

Assume, for the sake of contradiction, that the Toulouse accent is **not** the supremum of $\mathcal{A}$. Then, there exists an accent $a \in \mathcal{A}$ such that a is greater than the Toulouse accent.

However, upon hearing the Toulouse accent, any listener is irresistibly charmed and cannot help but prefer it over a. This contradicts our assumption that a is greater.

Thus, we conclude that our initial assumption is false, and the Toulouse accent must indeed be the supremum of $\mathcal{A}$. □

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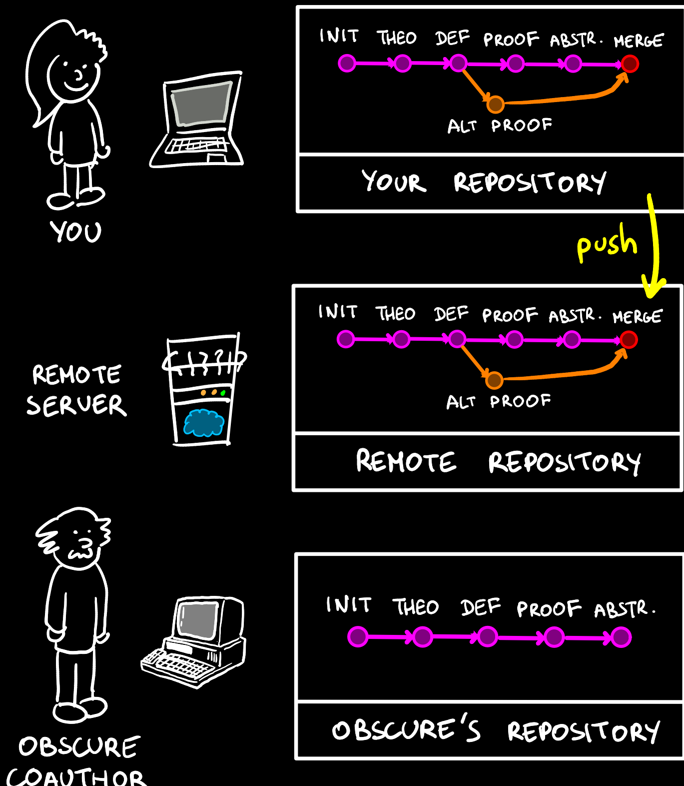

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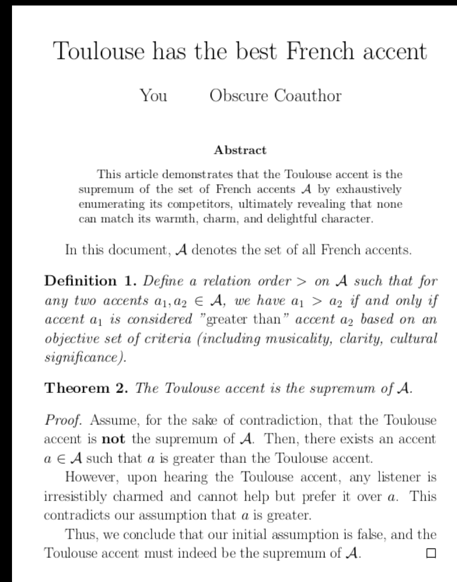

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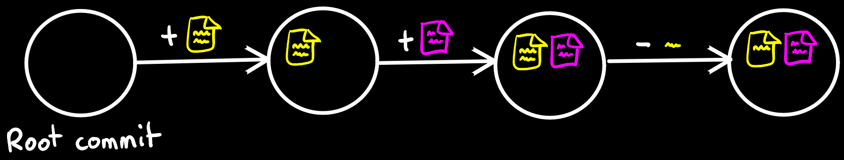
it stores all the project states over time.



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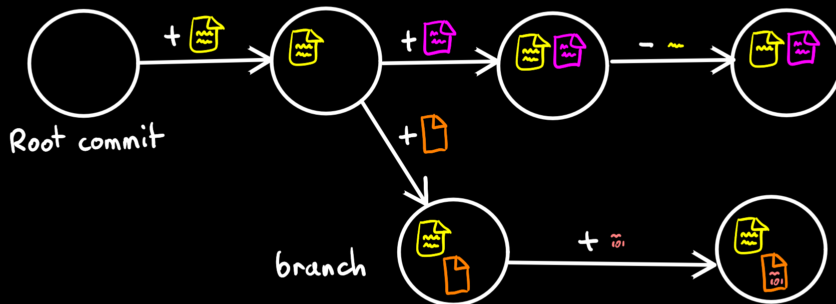

Git lets you create parallel development branches...

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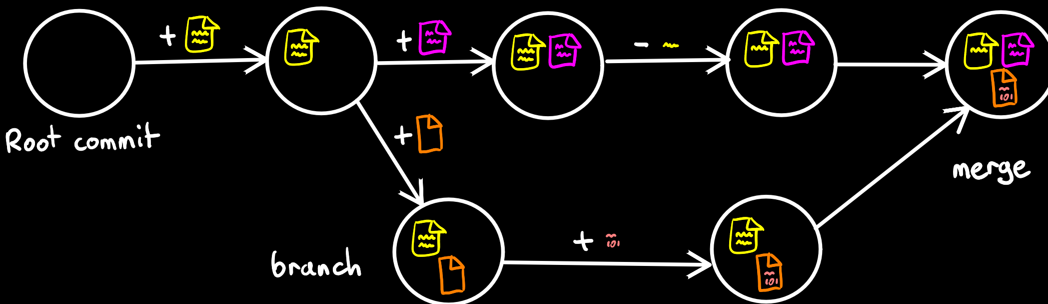

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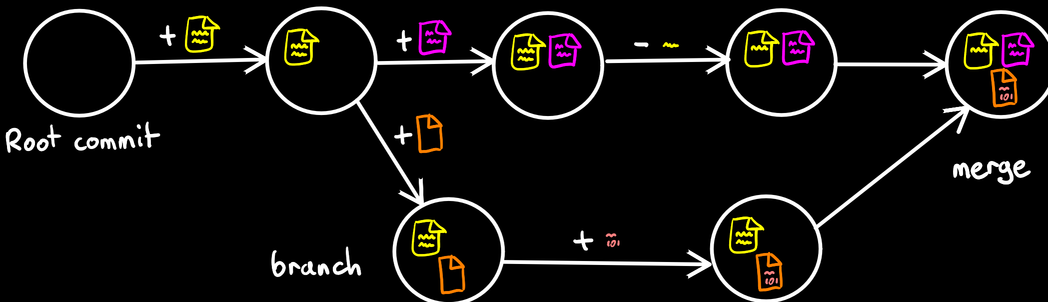

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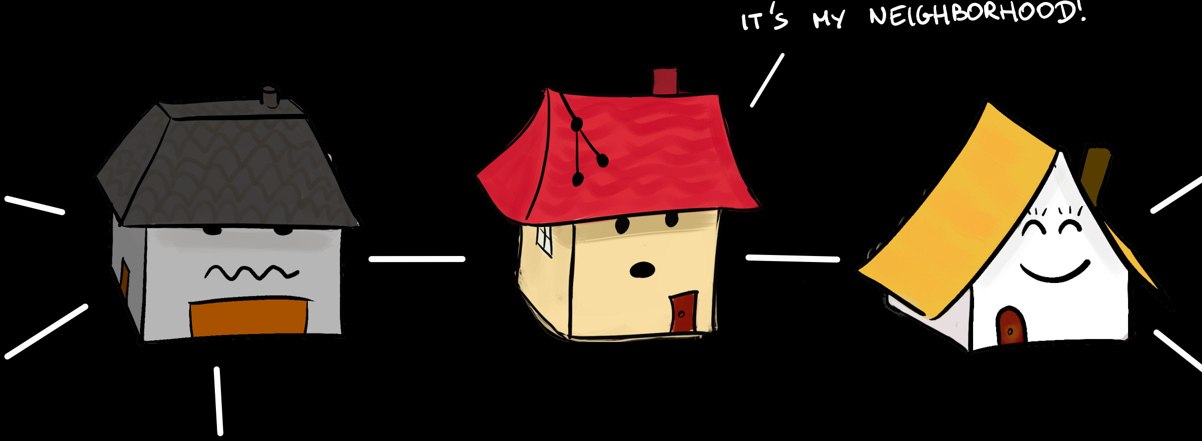
This forms a Directed Acyclic Graph (DAG), where the vertices are the project states, also named <u>commits</u>.

RANDOM GIT GRAPHS



OUR MOTIVATION

Objectives:
- Define a probabilistic model for graphs of commits
- Design random generators for such a distribution

①

Why?

②

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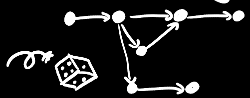
Why?

① VCS unit testing

use random graphs



to check program correctness

```
void test_remove_last_commit() {

 srand(time(NULL));

 for (int i = 0; i < NUM_TESTS; i++) {

  GitGraph graph1 = create_random_git_graph();
  size_t n1 = graph1.num_commits;

  GitGraph graph2 = remove_last_commit(graph1);
  size_t n2 = graph2.num_commits;

  assert(n1 == n2 + 1);

 }
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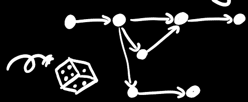

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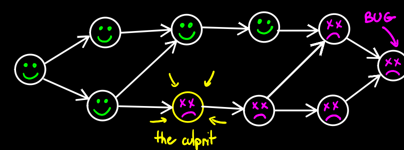
② Average-case complexity of VCS algorithms

+ benchmark

e.g.

→ **Regression Search Problem**

Find the commit that introduces a bug



Analysis of git bisect
[C. Dorbec Lecoq 2023]

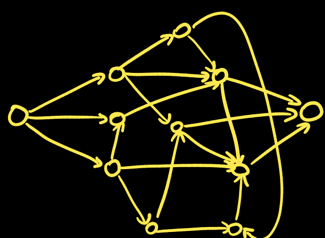
→ **reachability / label synchronisation Problem**
[Bulteau David Horn Tran-Girard]

WHICH GRAPHS TO CONSIDER ?

In git, every DAG

without restriction

can be generated...



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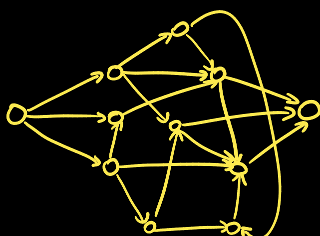

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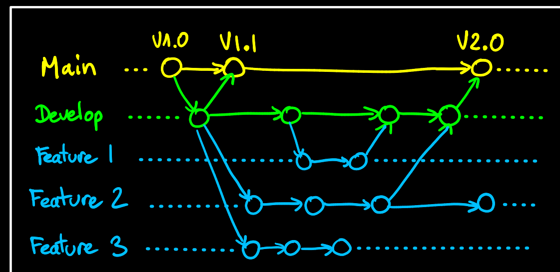

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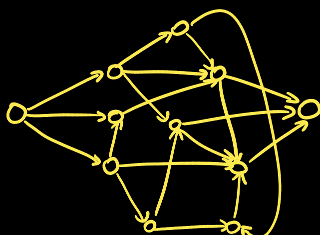

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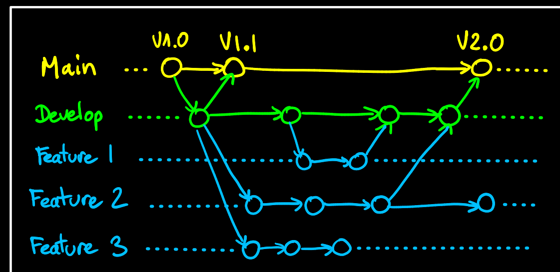


In the following, we consider a simple workflow but widely used in industry: the feature branch workflow

GIT GRAPH

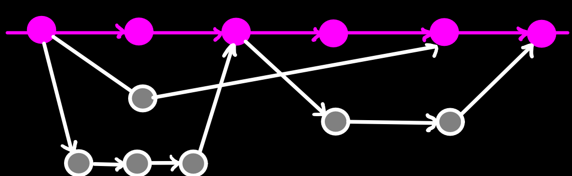
DEFINITION

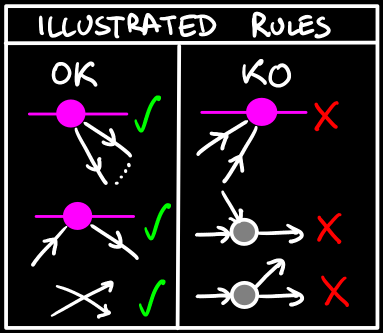
(feature branch) Git graph = DAG with

- a main branch (path of magenta vertices)
- 0, 1 or several feature branches, paths of ≥ 1 white vertices starting and ending on magenta vertices
- indegree ≤ 2 for all vertices

previously defined in [Lecoq 2024]

e.g.





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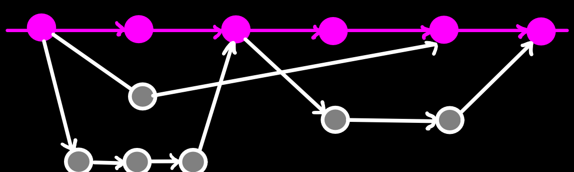
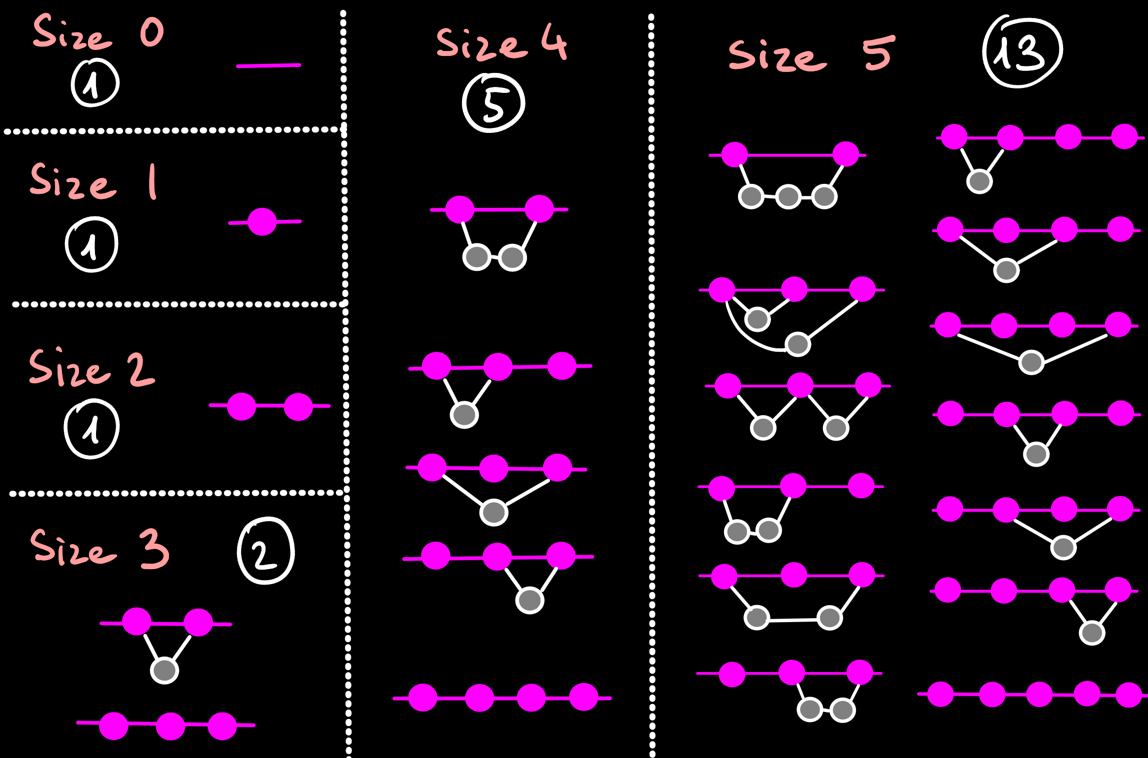


GOALS

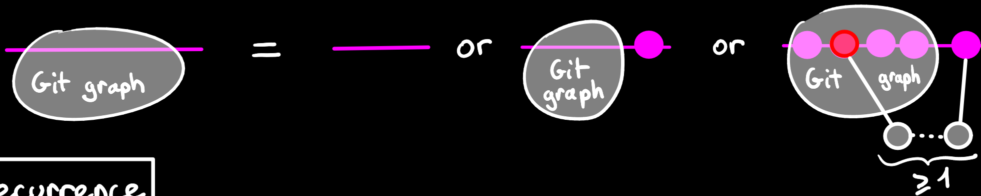
- Counting Git graphs (exactly or asymptotically)
- Random generation of Git graphs

ALL SMALL GIT GRAPHS



RECURSIVE DECOMPOSITION

Decomposition



Recurrence

$$g_{n,k} = g_{n-1,k-1} + \sum_{\ell \geq 1} (k-1)\, g_{n-1-\ell,\, k-1} \quad \text{for } n \geq 1$$

where $g_{n,k}$:= number of Git graphs with n vertices, k of them being magenta

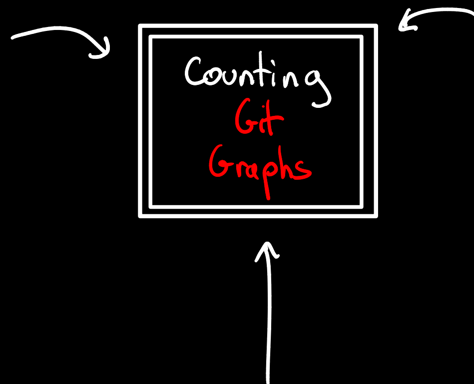
Differential Equation for the Generating Function

$$G(z,u) = 1 + zu\, G(z,u) + \frac{z^2 u^2}{1-z} \frac{\partial G}{\partial u}(z,u)$$

where $G(z,u) = \sum_{n \geq 0} \sum_{k \geq 0} g_{n,k}\, z^n u^k$

⚠ $G(z,u)$ is not analytic.

THREE ANGLES OF ATTACK



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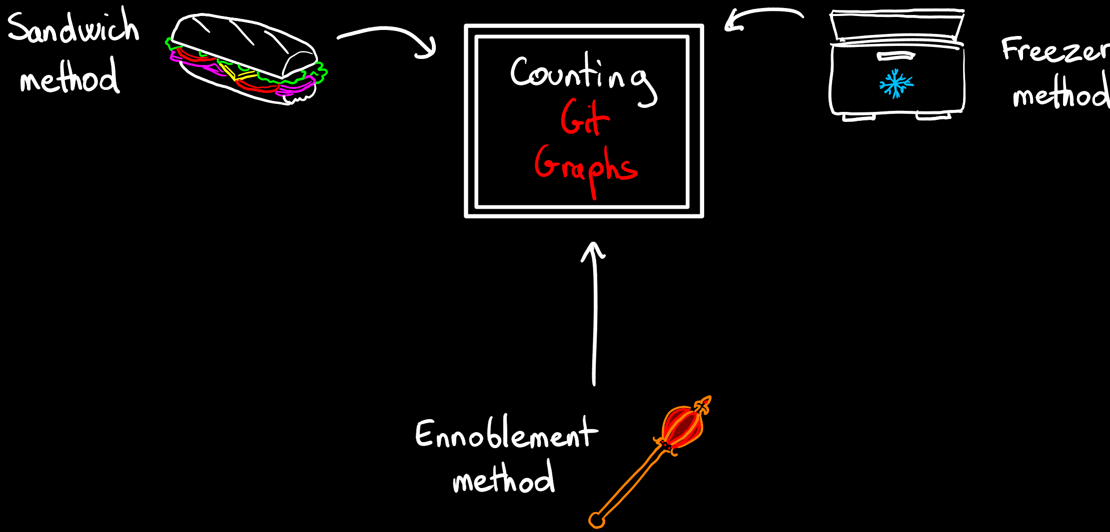

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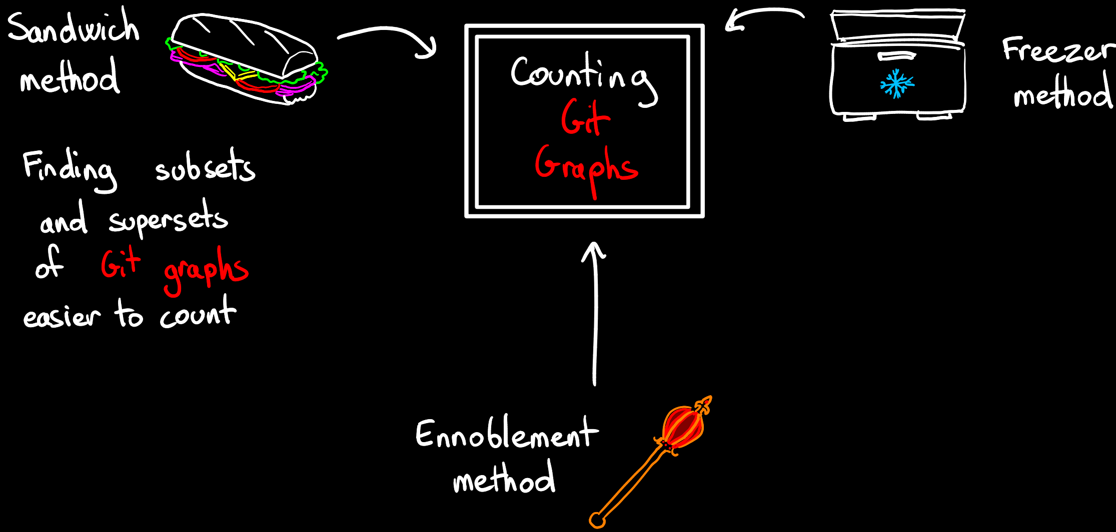
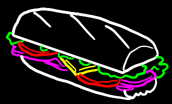


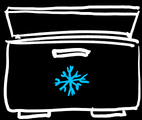
THREE ANGLES OF ATTACK

Sandwich method



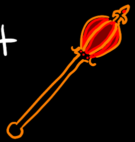
Finding subsets and supersets of Git graphs easier to count

Counting Git Graphs



Freezer method

Ennoblement method



making the generating function less ordinary

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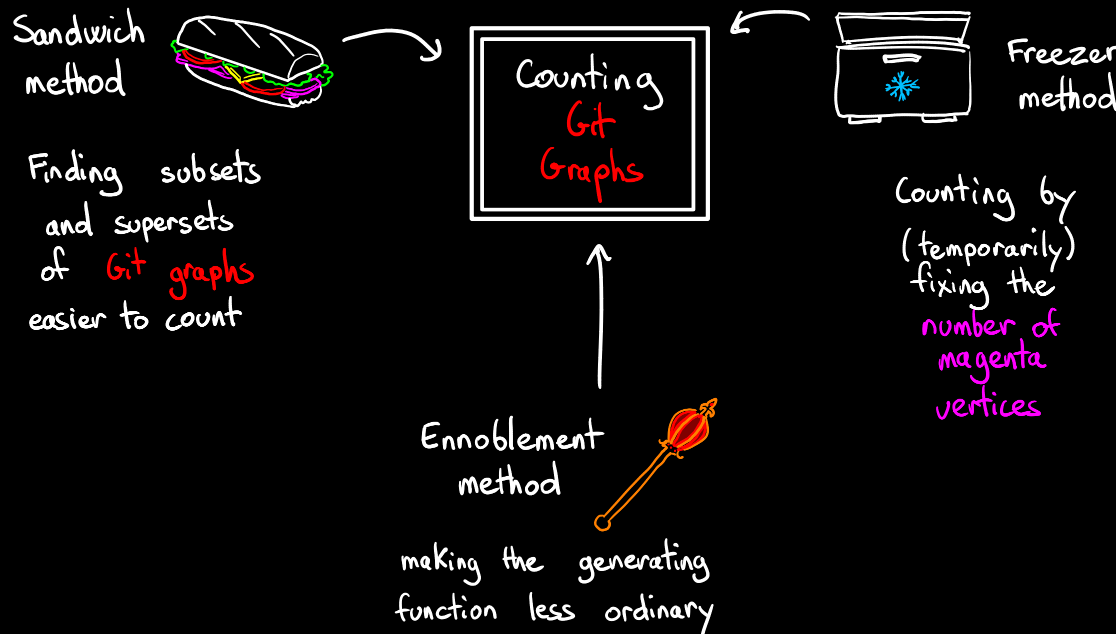

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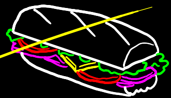


Finding subsets and supersets of Git graphs easier to count

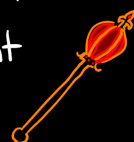
not developed today

Example of result

The random variable nb magenta vertices / n converges in probability to 1/2.

Counting Git Graphs

Ennoblement method



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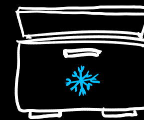

Counting by (temporarily) fixing the number of magenta vertices

THREE ANGLES OF ATTACK

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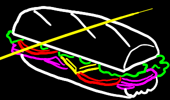

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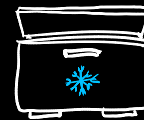

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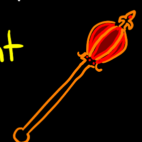

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Differential Equation	$G(z,u) = 1 + zu\,G(z,u) + \frac{z^2u^2}{1-z}\frac{\partial G}{\partial u}(z,u)$

Usual trick:

Ordinary Generating Function

$$\underbrace{\sum_{n,k \geq 0} g_{n,k} z^n u^k}_{G(z,u)},$$

not analytic ✗

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Exponential Generating Function $\underbrace{\sum_{n,k \geq 0} \frac{g_{n,k}}{n!} z^n u^k}$ analytic, but no pretty equation ✗

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Borel transform on u ↓

$\tilde{G}(z,u) = \sum_{n,k\geq 0} \frac{g_{n,k}}{k!} z^n u^k$ analytic ✓

and

Differential Equation for $\tilde{G}$
$\frac{\partial \tilde{G}}{\partial u} = z\tilde{G} + \frac{z^2u}{1-z}\frac{\partial \tilde{G}}{\partial u}$

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this can be solved!

Theorem

$$\tilde{G}(z,u) = \left(1 - \frac{z^2 u}{1-z}\right)^{-\frac{1-z}{z}}$$

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How can it be exploited?

DIFFERENT PERSPECTIVE

Is there a combinatorial explanation for the formula

$$\tilde{G}(z,u) = \left(1 - \frac{z^2 u}{1-z}\right)^{-\frac{1-z}{z}} \; ?$$

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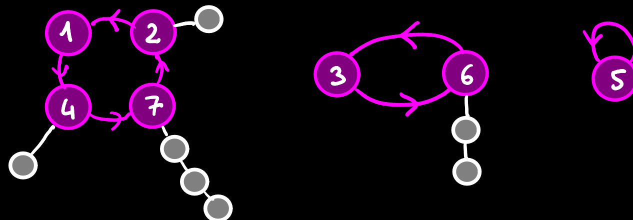
$$\tilde{G}(z,u) = \left(1 - \frac{z^2 u}{1-z}\right)^{-\frac{1-z}{z}} = \exp\left(\frac{1}{\frac{z}{1-z}} \ln\left(\frac{1}{1 - uz\frac{z}{1-z}}\right)\right) ?$$

DIFFERENT PERSPECTIVE

Definition

cyclarium = set of cycles of magenta vertices labeled from 1 to k where a chain of white unlabeled vertices is attached to each magenta vertex, except to the ones having the smallest label in their cycles.

e.g.:



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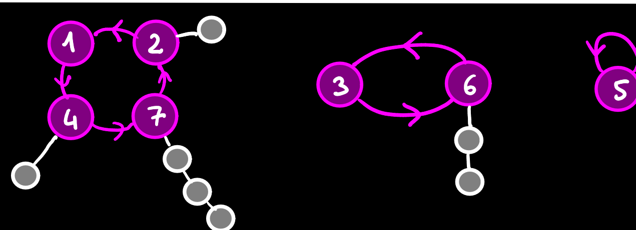

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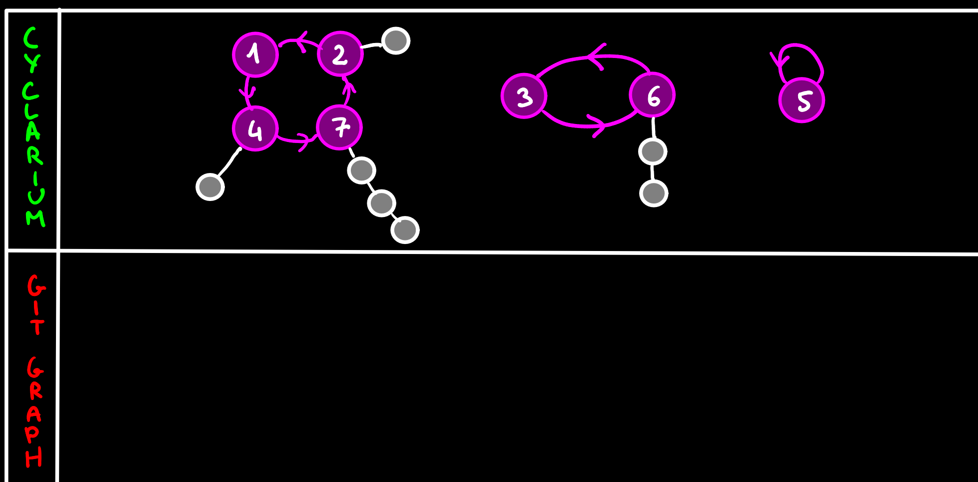
It's the generating function of cyclariums!

BIJECTION

Proposition

There is a bijection from cyclariums to Git graphs:



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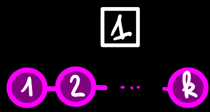

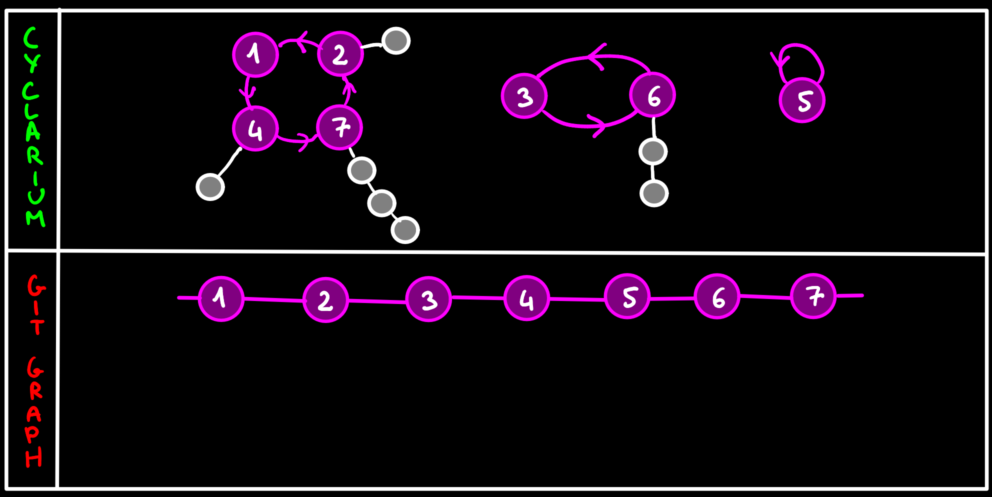

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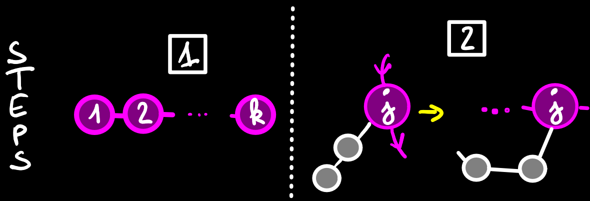

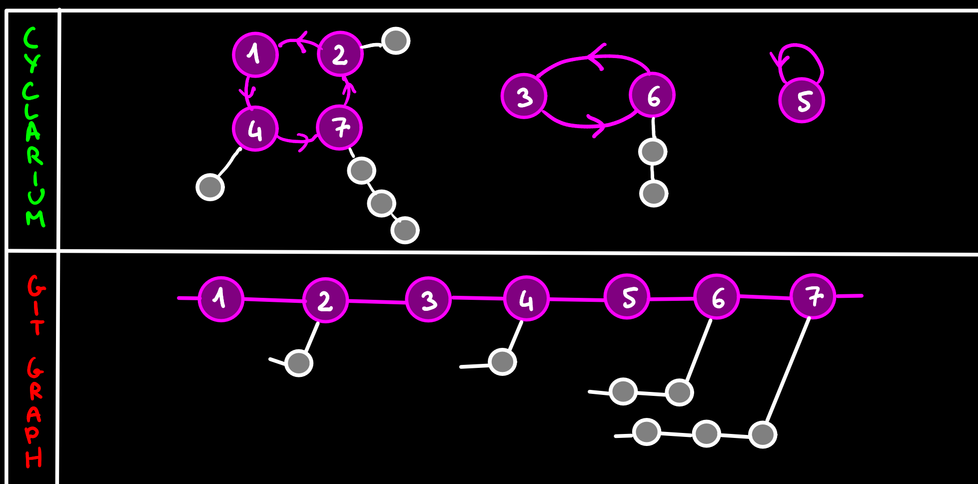

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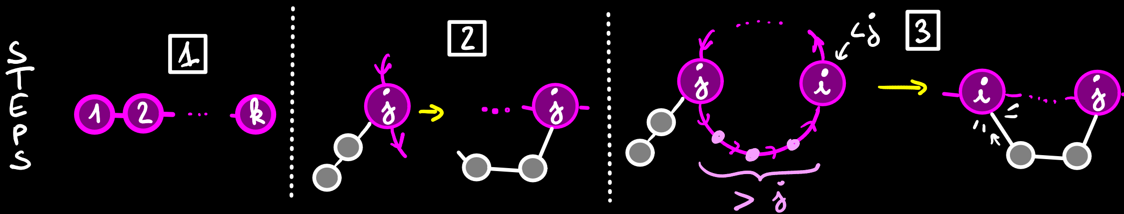

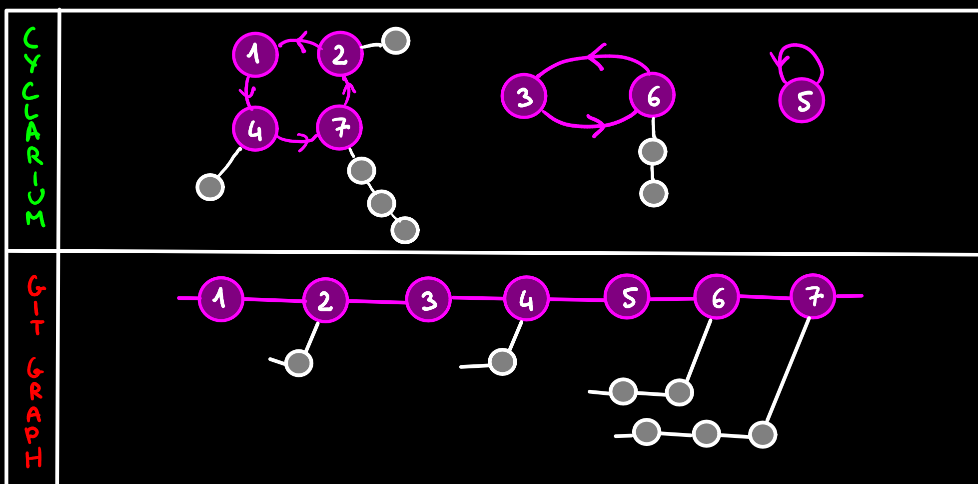

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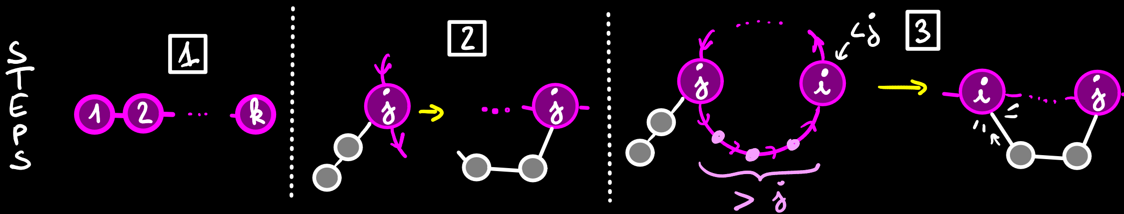

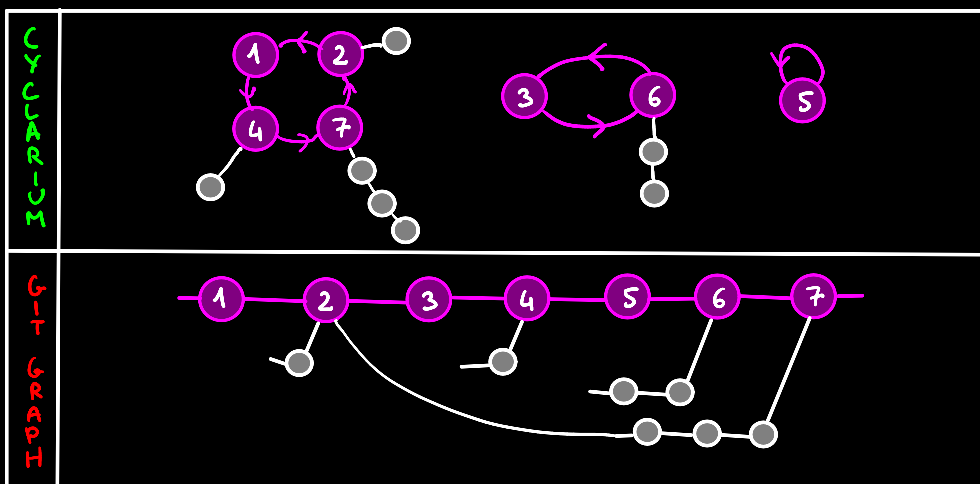

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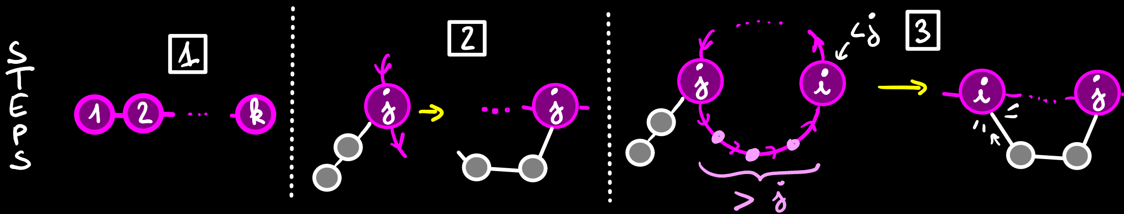

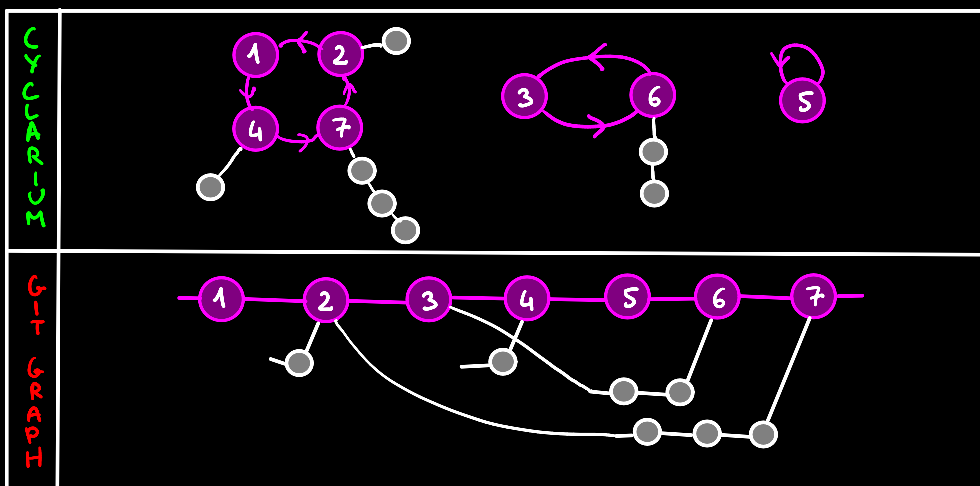

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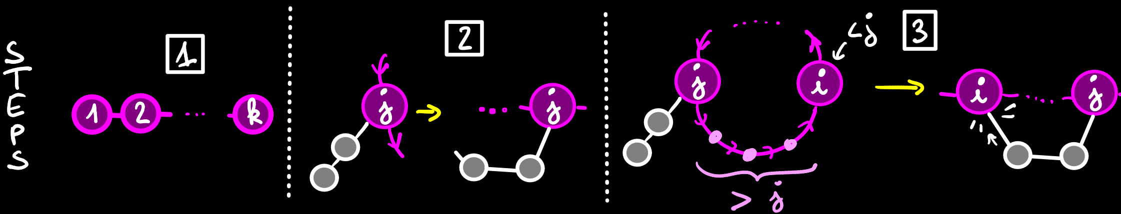

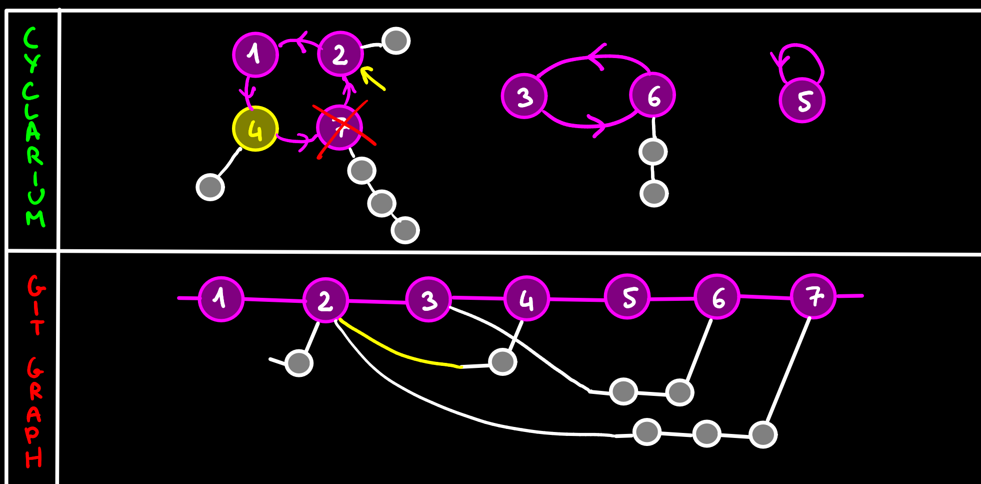

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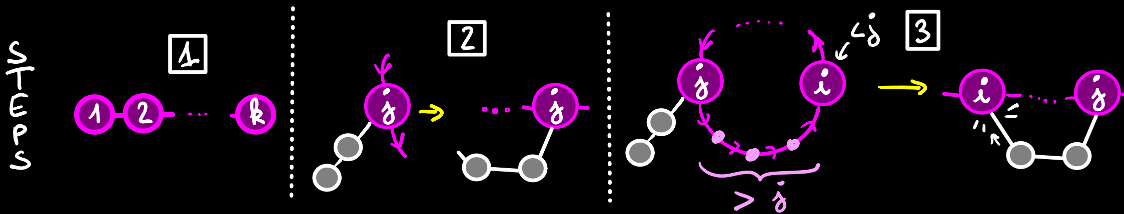

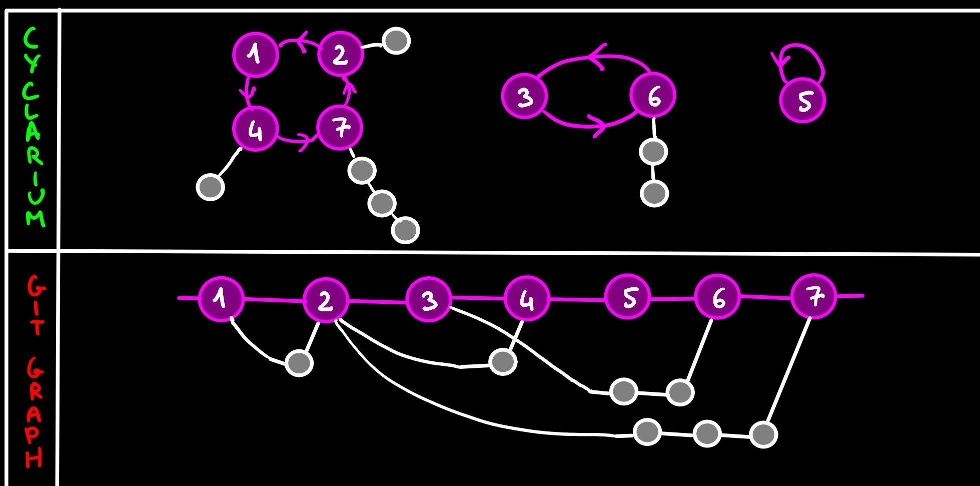

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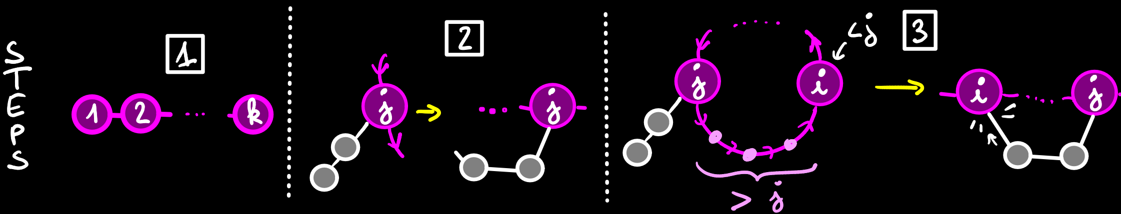

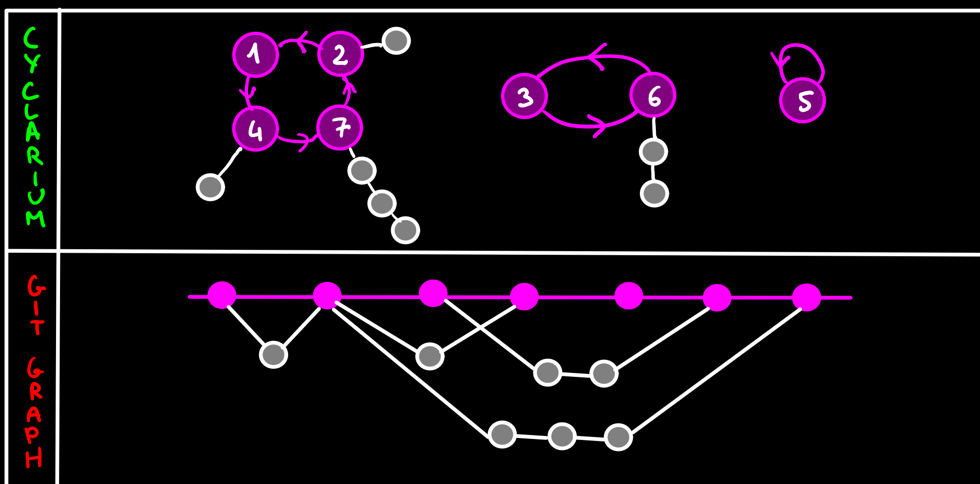

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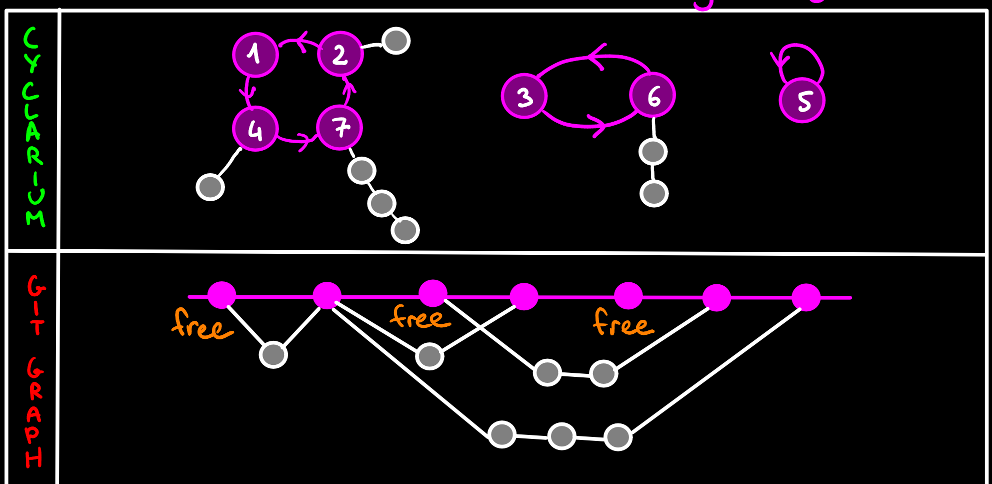
There is a bijection from cyclariums to Git graphs sending

vertices ⟶ vertices

magenta vertices ⟶ magenta vertices

cycles ⟶ free vertices
i.e magenta vertices of indegree ≤ 1

cycle lengths ⟶ magenta vertices in the connected components when the magenta edges are erased.



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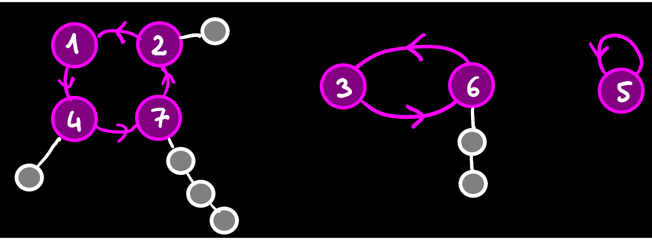
CYCLARIUM



Corollary

$$g_{n,k} = \sum_{f=1}^{k-1} \left[\begin{matrix} k \\ f \end{matrix} \right] \binom{n-k-1}{k-f-1} \qquad (k < n)$$

where $g_{n,k}$ = number of Git graphs counted by vertices & magenta vertices

and $\left[\begin{matrix} \cdot \\ \cdot \end{matrix} \right]$ = (unsigned) Stirling number of 1st kind

RANDOM GENERATION

THE RANDOM GENERATOR WE WANT

Inputs: Size n and number of magenta vertices k

Output: A random Git graph of size $\simeq n$ and $\simeq k$ magenta vertices

(uniformly when conditioned to have size n and k magenta vertices)

RANDOM MODEL

"Boltzmann model" (exponential in u, ordinary in z)

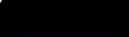
Fix $z > 0$ and $u > 0$.

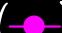
We wish to draw a Git graph γ with a weight proportional to

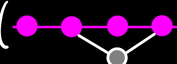
$$z^{\#\text{vertices in } \gamma} \; \frac{u^{\#\text{magenta vertices in } \gamma}}{(\#\text{magenta vertices in } \gamma)!}$$

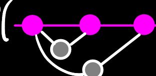
(Size is not fixed)

Examples

$\mathbb{P}($  $) \propto 1$

$\mathbb{P}($  $) \propto z\,u$

$\mathbb{P}($  $) \propto z^5 \frac{u^4}{24}$

$\mathbb{P}($  $) \propto z^5 \frac{u^3}{6}$

RANDOM MODEL

"Boltzmann model" (exponential in u, ordinary in z)

Fix $z > 0$* and $u > 0$.*

We wish to draw a Git graph γ with a weight ~~proportional~~ equal to

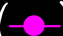
$$\frac{z^{\#\text{vertices in }\gamma}}{\tilde{G}(z,u)} \frac{u^{\#\text{magenta vertices in }\gamma}}{(\#\text{magenta vertices in }\gamma)!}$$

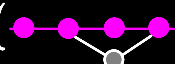
(Size is not fixed)

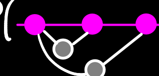
where $\tilde{G}(z,u) = \sum_{n,k \geq 0} \frac{g_{n,k}}{k!} z^n u^k = \left(1 - \frac{z^2 u}{1-z}\right)^{-\frac{1-z}{z}}.$

Examples

$\mathbb{P}($  $) = \frac{1}{\tilde{G}(z,u)}$

$\mathbb{P}($  $) = \frac{z\,u}{\tilde{G}(z,u)}$

$\mathbb{P}($  $) = \frac{z^5}{\tilde{G}(z,u)} \frac{u^4}{24}$

$\mathbb{P}($  $) = \frac{z^5}{\tilde{G}(z,u)} \frac{u^3}{6}$

*: in the disk of convergence of $\tilde{G}$

HOW TO SAMPLE GIT GRAPHS

Inputs : Size n and number of magenta vertices k

Step 1 Solve numerically z and u such that :

$$\mathbb{E}_{\text{boltz}(z,u)}(\text{size}) = n \qquad \mathbb{E}_{\text{boltz}(z,u)}(\#\text{mag}) = k$$

Step 2 Use a Boltzmann sampler for cyclariums for parameters z and u

Step 3 Bijection to Git graphs

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$$\mathbb{E}_{\text{boltz}(z,u)}(\text{size}) = \underbrace{z \frac{\partial}{\partial z} \ln \tilde{G}(z,u)}_{\text{computable}} \qquad \mathbb{E}_{\text{boltz}(z,u)}(\#\text{mag}) = \underbrace{u \frac{\partial}{\partial u} \ln \tilde{G}(z,u)}_{\text{computable}}$$

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Step 2 Use a Boltzmann sampler for cyclariums for parameters z and u

Step 3 Bijection to Git graphs

Fun fact

Let $\alpha = k/n$

if $\alpha < \frac{1}{2}$, $z \to \mathrm{fct}(\alpha)$ $u \to \mathrm{fct}(\alpha)$

if $\alpha > 1/2$, $z \sim \frac{\mathrm{fct}(\alpha)}{n}$, $u \sim \mathrm{fct}(\alpha)\, n^2$

ASYMPTOTIC ESTIMATE

ASYMPTOTIC ESTIMATE

Theorem The number of Git graphs with n vertices is asymptotically equivalent to

$$g_n \sim \frac{e^{1/8}}{2}\left(\frac{n}{2e}\right)^{\frac{n}{2}} \exp\left(\frac{1}{2}\ln\left(\frac{n}{2}\right)\sqrt{\frac{n}{2}} + \sqrt{2n} + \frac{\left(\ln\frac{n}{2}\right)^2}{32}\right)\left(\frac{n}{2}\right)^{-\frac{3}{8}}$$

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How to get such a result?

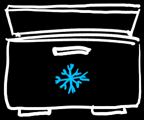
Freezer method !

IT'S EASIER WHEN YOU FIX k

$$G_k(z) := \sum_{n \geqslant 0} g_{n,k} z^n$$

= Generating Function of Git graphs where the number k of magenta vertices is fixed

Freeze k



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Freeze k

Claim

$$G_k(z) = z^k \prod_{j=0}^{k-1} \left(1 + \frac{j z}{1-z}\right)$$

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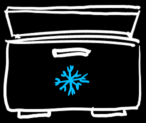

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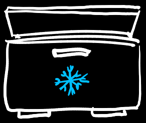

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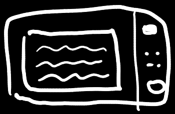
Cauchy integral formula

$$g_{n,k} = \frac{1}{2\pi i} \oint \frac{z^{2k-n-1}}{(1-z)^k} \frac{\Gamma\left(k + (1-z)/z\right)}{\Gamma\left((1-z)/z\right)}\, dz$$

Unfreeze k



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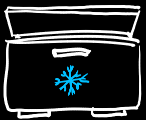
$$G_k(z) := \sum_{n \geq 0} g_{n,k} z^n$$

= Generating Function of Git graphs where the number k of magenta vertices is fixed

Freeze k



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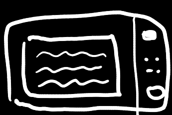
Cauchy integral formula

Unfreeze k



$$g_{n,k} = \frac{1}{2\pi i} \oint \frac{z^{2k-n-1}}{(1-z)^k} \frac{\Gamma(k + (1-z)/z)}{\Gamma((1-z)/z)} dz$$

→ invitation to saddle-point ...

SADDLE POINT ANALYSIS

$$g_{n,k} = \frac{1}{2\pi i} \oint \frac{z^{2k-n-1}}{(1-z)^{k}} \frac{\Gamma(k + z/(1-z))}{\Gamma(z/(1-z))} dz$$

We use saddle-point method on g_{n,k_n} where $k_n = \frac{n}{2} + \alpha_n \sqrt{\frac{n}{2}}$

and α_n sub-polynomial

(an educated guess about the location of the maximum)

$\downarrow$

horrible formula for g_{n,k_n}

SADDLE POINT ANALYSIS

$$g_{n,k} = \frac{1}{2\pi i} \oint \frac{z^{2k-n-1}}{(1-z)^k} \frac{\Gamma\left(k + z/(1-z)\right)}{\Gamma\left(z/(1-z)\right)} dz$$

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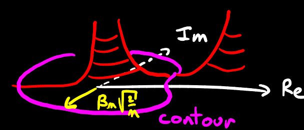
(an educated guess about the location of the maximum)

$$g_{n,k_n} \sim \left(\frac{n}{2e}\right)^{\frac{n}{2}} \exp\left(\lambda_n(\beta_n)\sqrt{\frac{n}{2}} + \mu_n(\beta_n)\right) \left(\frac{2}{n}\right)^{\frac{1}{4}} \sqrt{\frac{\beta_n}{\pi \ln\left(\frac{n}{2}\right)}}$$

where $\beta_n \sqrt{\frac{2}{n}}$ is the saddle-point



$$\lambda_n(x) = \frac{1}{2x}(1+x)\ln\left(\frac{n}{2}\right) + \frac{1}{x} + x + \frac{\ln(x)}{x} - x\ln(x) + \frac{\ln(2)}{x^2}$$

$$\mu_n(x) = \frac{\ln\left(\frac{n}{2}\right)^2}{32x^2} + \left(\frac{1}{4x^2} - \frac{5}{8} + \frac{3}{8}\frac{\ln(x)}{x^2} - \frac{1}{4}\ln(x)\right)\ln\left(\frac{n}{2}\right) + \frac{1}{2x^2} - \frac{1}{2} + \frac{x}{8} + \frac{5}{8}\frac{\ln(x)^2}{x^2} - \frac{\ln(x)^2 x}{2} + \frac{3}{2}\frac{\ln(x)^2}{x^2} - \frac{1}{2}\ln(x)^2 - \frac{1}{4}\ln(x)$$

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Disclaimer: not totally proved

SOME LIMIT LAWS WE SHOULD OBTAIN UNDER UNIFORM DISTRIBUTION

→ Number of magenta vertices converges to a Gaussian distribution of mean $\sim \frac{n}{2} + \frac{1}{4}\ln\left(\frac{n}{2}\right)\sqrt{\frac{n}{2}} - \sqrt{2n}$ (local limit)

Variance to be determined...

→ Number of free vertices converges to a Gaussian distribution of mean $\sim \frac{1}{2}\sqrt{\frac{n}{2}}\ln\left(\frac{n}{2}\right)$ (distribution)

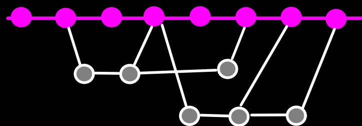
and variance $\sim \frac{1}{2}\sqrt{\frac{n}{2}}\ln\left(\frac{n}{2}\right)$

PERSPECTIVES ABOUT RANDOM GIT GRAPHS

→ Other random models

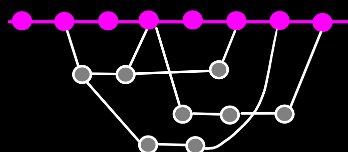
- Other graph models ▷ Collaboration with Clement + Maréchal + Pépin

Phoenix graphs



merged branches can be reborn

Fork Anywhere graphs



branches can be born anywhere but must be merged into main

More involved workflows

- more probabilistic models [Nicaud Peyraud-Magnin Rotondo]

Thank you!